No. 5505 Engineer's Computation Pad

$$\operatorname{div}\bar{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial A_z}{\partial z}$$ Divergence of a vector field

$$\operatorname{div}\bar{E} = \frac{\rho}{\varepsilon}$$ is ε is constant (Gauss' law differential)

del operator $$\bar{\nabla} \equiv \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

$$\bar{\nabla}\cdot\bar{A} \equiv \operatorname{div}\bar{A}$$ (del dot A)

$$\bar{\nabla}\cdot\bar{E} = \operatorname{div}\bar{E} = \rho/\epsilon \qquad (\bar{D} = \epsilon\bar{E})$$

$$\oint_{\text{surf.}} \bar{D}\cdot d\bar{S} = \oint \epsilon\bar{E}\cdot d\bar{S} = \int_{\text{volume}} \rho\, du = Q_{\text{enclosed}}$$

(du = differential volume element)

$$\rho = \bar{\nabla}\cdot\epsilon\bar{E}$$

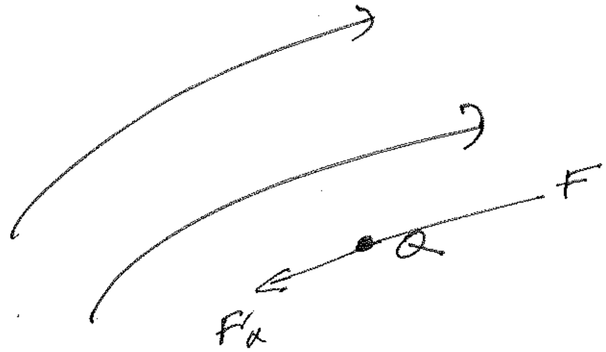
Green's theorem is 3-d (divergence theorem)

$$\oint_{\text{surface}} \bar{D}\cdot d\bar{S} = \int_{\text{volume}} (\bar{\nabla}\cdot\epsilon\bar{E})\, du$$

$$\bar{\nabla}\cdot K\bar{A} = \bar{A}\cdot\bar{\nabla}K + K(\bar{\nabla}\cdot\bar{A})$$

K scalar field (like the potential)

ENERGY + ELECTRIC POTENTIAL



In an electric field $\bar{E}$ a point charge experiences a force $\bar{F} = Q \cdot \bar{E}$

This ~~fo~~ unbalanced force will accelerate the charge and it's motion will be along the direction of the field if Q is positive

$$F_a = -\bar{F}_Q = -Q\bar{E}$$

to balance the F_Q

<u>WORK</u> is defined as a force acting over distance. The differential amount of work dW is done by the applied force when the charged particle moves through a differential distance dl

Work maybe + or − depending on the direction $d\bar{l}$ the vector displacement with respect to the applied force $\bar{F}_a$ when $d\bar{l}$ and $\bar{F}_a$ are not in the same direction

$$dW = F_a\, dl \cos\theta = \bar{F}_a \cdot d\bar{l}$$

$$W = -Q\oint \bar{E}\cdot d\bar{l}$$

$$d\bar{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$
$$d\bar{l} = dr\,\hat{a}_r + r\,d\varphi\,\hat{a}_\varphi + dz\,\hat{a}_z$$
$$d\bar{l} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\varphi\,\hat{a}_\varphi$$

The work done in moving a point charge Q from B to A in a static field is the same for any path.
Equivalently the work done in moving around a charge in any closed loop is zero

$$\oint \bar{E}\cdot d\bar{l} = 0 \quad \text{(static fields)}$$

Such a vector field is conservative

ELECTRIC POTENTIAL

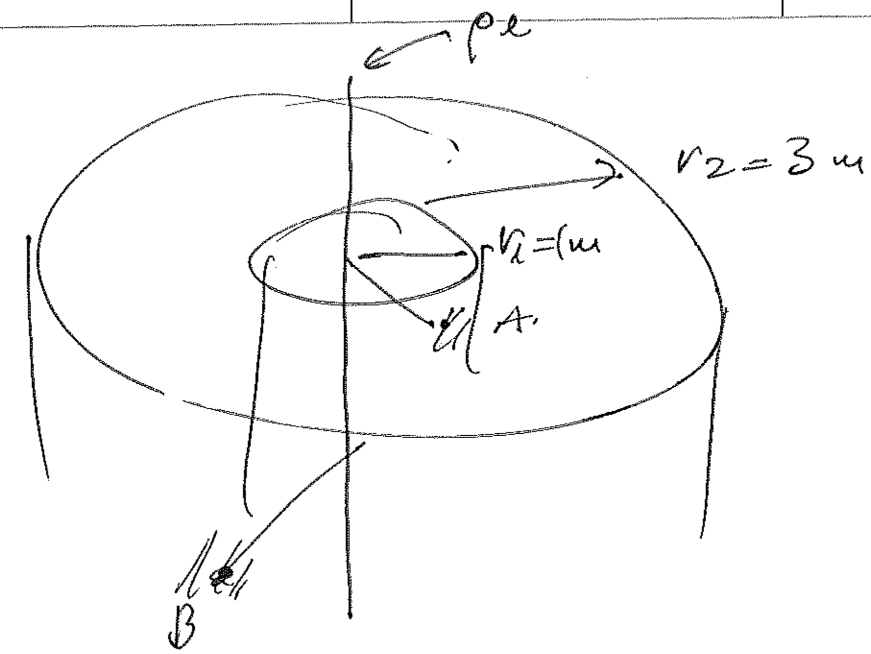
The potential of point A with respect to point B is defined as the work done in moving a unit positive charge Q_u from B to A

$$V_{AB} = \frac{W}{Q_u} = -\int_{B}^{A} \bar{E}\cdot d\bar{l} \quad (V)$$

initial point is the reference
lower limit of the line integral

$\bar{E}$ = conservative $\Rightarrow$

$$V_{AB} = V_{AC} - V_{BC}$$



Find the potential of A $(1, \varphi, 2)$ with respect to B $(3, \varphi', 2')$ in cylindrical coordinates where the electric field due to ~~the~~ a line charge on the z axis is given by

$$\bar{E} = \left(\frac{50}{r}\right) \hat{a}_r \ \frac{V}{m}$$

$d\bar{l}$ $(\hat{a}_r, \hat{a}_\varphi, \hat{a}_z)$ but $\bar{E}$ has only $\hat{a}_r$

so $\bar{E} \cdot d\bar{l} = E_r dr$

$$V_{AB} = -\int_B^A \bar{E} \cdot d\bar{l} = -\int_3^1 \frac{50}{r} dr$$

$$= -50 \ln \frac{1}{3} = 55 \ V$$

poin A is @ higher potential than point B

Because no work is done in motion along $\hat{a}_\varphi$ or $\hat{a}_z$ all points on the cylinder $r = ct$ must be @ same potential

POTENTIAL OF POINT CHARGE

$\bar{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{a}_r$ only radial

$$V_{AB} = -\int_B^A \bar{E}\cdot d\bar{l}$$

$$= -\int_{r_B}^{r_A} E_r dr =$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_{r_B}^{r_A}\frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

For positive Q point A is @ higher potential than point B when $r_A < r_B$

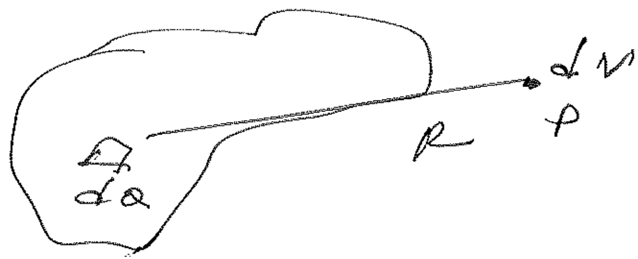
The equipotential surfaces are concentric spherical shells

$\boxed{\text{send B to infinity}}$ (reference)

$$V_{A\infty} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

DANGER do not forget who is in infinity

POTENTIAL OF A CHARGE DISTRIBUTION

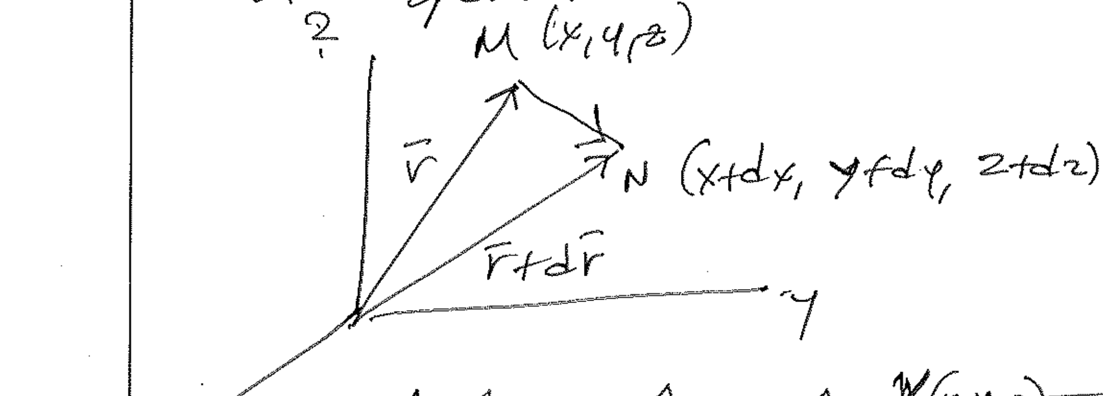


$$dV = \frac{dQ}{4\pi\epsilon_0 R}$$ integrate over the volume

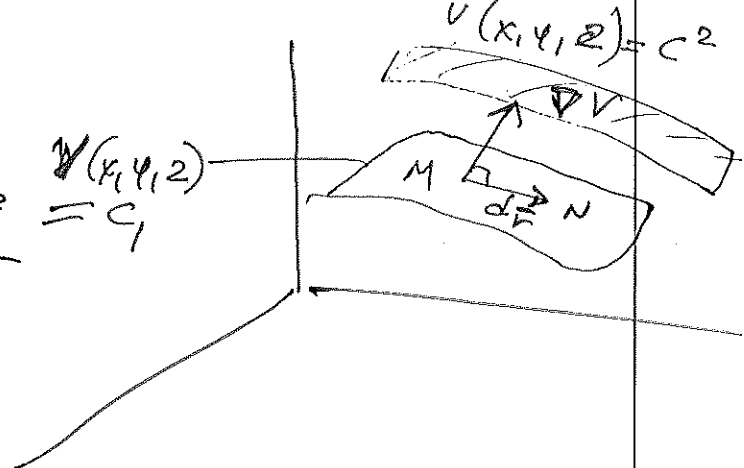
$$V = \int_{vol} \frac{\rho\, dv}{4\pi\epsilon_0 R}$$

GRADIENT

Another operator



$$d\bar{r} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$



$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

the del operator operating on V

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

$$dV = \bar{\nabla} V \cdot d\bar{r}$$

the vector field $\bar{\nabla} V$ (grad V) is called the gradient of the scalar function V. For fixed $|d\bar{r}|$ the change in V in a given direction $d\bar{r}$ is proportional to the projection of ∇V in that direction.

∇V lies in the direction of maximum increase of the function V

If M, N lie in the same equipotential then $dV = 0$ and $\bar{\nabla} V$ is $\perp$ $d\bar{r}$
$d\bar{r}$ is tangent to the equipotential

$\bar{\nabla} V$ is along increased V from
$V(x, y, z) = c_1$ $V(x, y, z)$ c_2 $c_2 > c_1$

The gradient of a potential function is a vector field that is everywhere normal to the equipotential surfaces

$$\bar{\nabla} V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\bar{\nabla} V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{\partial V}{r\partial \varphi}\hat{a}_\varphi + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\bar{\nabla} V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{\partial V}{r\partial \theta}\hat{a}_\theta + \frac{\partial V}{r \sin\theta\, \partial \varphi}\hat{a}_\varphi$$

(del is only defined with cartesian)

* Relation between $\bar{E}$ and V

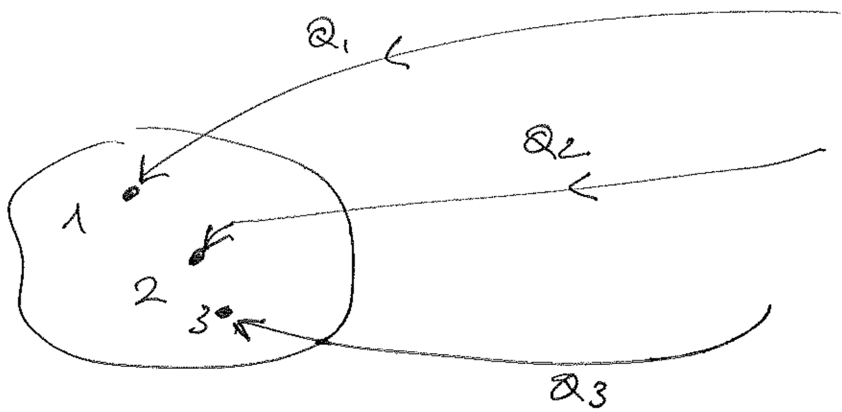
$$dV = -\bar{E}\cdot d\bar{l}$$
$$dV = \bar{\nabla} V \cdot d\bar{r}$$

$$\bar{E} = -\bar{\nabla} V$$

The electric field intensity $\bar{E}$ maybe obtained when the potential function is known by simply taking the negative gradient of V: The gradient is found to be a vector normal to the equipotential surfaces directed to a positive change in V. With the $-$ sign above $\bar{E}$ is found to be directed from higher to lower V

FEB 14

Energy in static E field



Consider the work done to assemble charge by charge a distribution $n=3$ point charges
The region is assumed initially to be charge free and with $\bar{E}=0$ throughout

to bring $Q_1 \longrightarrow 1$ $\quad W_1 = 0 \quad$ (no $\bar{E}$)

to bring $Q_2 \longrightarrow 2$ $\quad W_2 = Q_2 \cdot V_{2,1}$

the potential @ point 2 due to charge 1

to bring $Q_3 \rightarrow 3$ $\quad W_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$

$$\overline{W_E} = 0 + W_2 + W_3 = 0 + Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2}$$

If the charges were put in reverse order

$$\underline{W_E} = \underset{\substack{\| \\ 0}}{W_3} + \underset{\substack{\| \\ W_2}}{Q_2 V_{3,2}} + Q_1 V_{1,3} + Q_1 V_{1,2}$$

$$2W_E = \underbrace{Q_1 (V_{1,2} + V_{1,3})}_{Q_1 \cdot V_1} + Q_2(\underbrace{V_{2,1} + V_{2,3}}_{V_2}) + Q_3(\underbrace{V_{3,1} + V_{3,2}}_{V_3})$$

V_1 is the potential @ P1 $\qquad 2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3$

$$W_E = \frac{1}{2}\left(Q_1V_1 + Q_2V_2 + Q_3V_3\right) = \frac{1}{2}\sum_{m=1}^{n} Q_m V_m$$

for a region containing ~~p~~ n point charges

for a region w/ charge density ρ (C/m^3)
the summation becomes integration

$$W_E = \frac{1}{2}\int \rho V\, dv$$

other forms

$$\left.\begin{aligned} W_E &= \frac{1}{2}\int \bar{D}\cdot\bar{E}\, dv \\ W_E &= \frac{1}{2}\int \epsilon E^2\, dv \\ W_E &= \frac{1}{2}\int \frac{D^2}{\varepsilon}\, dv \end{aligned}\right\} \text{same}$$

In an electric circuit the energy stored in the field of a capacitor is given by

$$W_E = \frac{1}{2} Q V^2 = \frac{1}{2} C V^2$$

where C is the capacitance in Farads and V is the votage difference between 2 conductors making up the capacitor an Q is the magnitude of total charge in one of the conductors

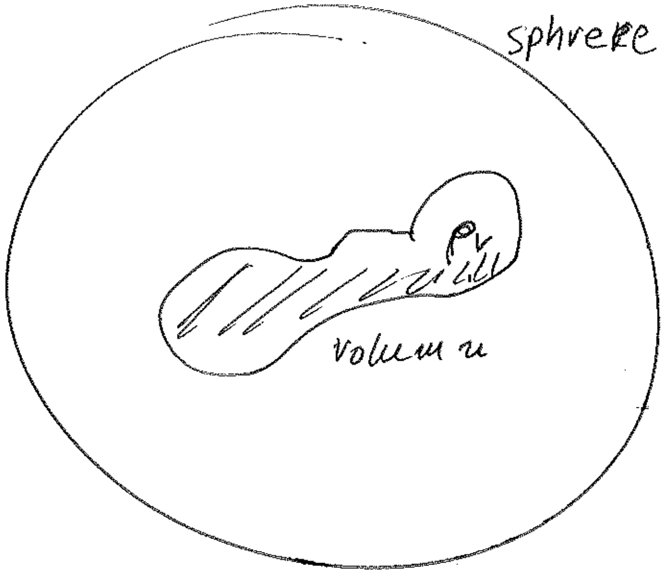
$\boxed{\epsilon \bar{E} = \bar{D}}$

Prove that

(11)

$$W_E = \frac{1}{2}\int \epsilon E^2 du \quad = \cancel{\frac{1}{2}\int}$$

Charge distributed through a volume u with density ρ gives rise to an electric field with $W_E = \frac{1}{2}\int_u V\rho\, du$ or $W_E = \frac{1}{2}\int \epsilon E^2 du$



$$\oint \bar{D} \cdot d\bar{S} = \int \rho dV = Q_{enc}$$

$$\rho = \bar{\nabla} \cdot \bar{D}$$

$$\oint \bar{D} \cdot d\bar{S} = \int (\bar{\nabla} \cdot \bar{D}) du$$

charge-containing voulum u enclosed within large sphere R, $\rho = 0$ outside u

$$W_E = \frac{1}{2}\int_u \rho V du = \frac{1}{2}\underbrace{\int}_{\text{spherical volume}} \rho V du = \frac{1}{2}\underbrace{\int}_{\text{sperical volume}} (\bar{\nabla} \cdot \bar{D}) V du$$

$$\nabla \cdot V\bar{A} = \bar{A} \cdot \bar{\nabla} V + V(\bar{\nabla} \cdot \bar{A}) \Rightarrow$$

$$\Rightarrow V(\bar{\nabla} \cdot \bar{A}) = \underline{\bar{\nabla} \cdot V\bar{A}} - \bar{A} \cdot \bar{\nabla} V$$

$$= \cancel{\frac{1}{2}\int} \quad = \frac{1}{2}\int (\bar{\nabla} \cdot V\bar{D}) du - \frac{1}{2}\int (\bar{A} \cdot \bar{\nabla} V) du$$

$$W_E = \frac{1}{2}\int_{\text{spere}} (\bar{\nabla}\cdot V\bar{D})\,du - \frac{1}{2}\int_{\text{spher}} (\bar{D}\cdot\bar{\nabla}V)\,du$$

take $R \to \infty$

divergence theorem

$$\oint_{\text{surface}} V\bar{D}\cdot d\bar{S}$$

take $R \to \infty$ then the enclosed volume looks like a point charge. At the surface

$$\left.\begin{array}{ll} \bar{D} \text{ appears} \sim \dfrac{K_1}{R^2} \\ V \text{ appe} \sim \dfrac{K_2}{R} \end{array}\right\} \text{ integrant } \frac{1}{R^3}$$

$$\lim_{R\to\infty}\left(\oint_{\substack{\text{surface} \\ \cancel{R\to\infty}}} V\bar{D}\cdot d\bar{S} = \right) \qquad dS \to R^2$$

$$\lim_{R\to\infty}\left(\frac{R^2}{R^3}\right) = 0$$

No. 5505 Engineer's Computation Pad

$\bar{D} = \epsilon \bar{E}$ | $\bar{E} = -\bar{\nabla} V$

$-\frac{1}{2}\int (\bar{D} \cdot \bar{\nabla} V) \, du =$ (13)

$= \frac{1}{2}\int \epsilon \bar{E} \cdot (-\bar{E}) \, du = +\frac{1}{2}\int \epsilon \bar{E}^2 \, du$

sphere $R \to \infty$

~~spherical~~ volume

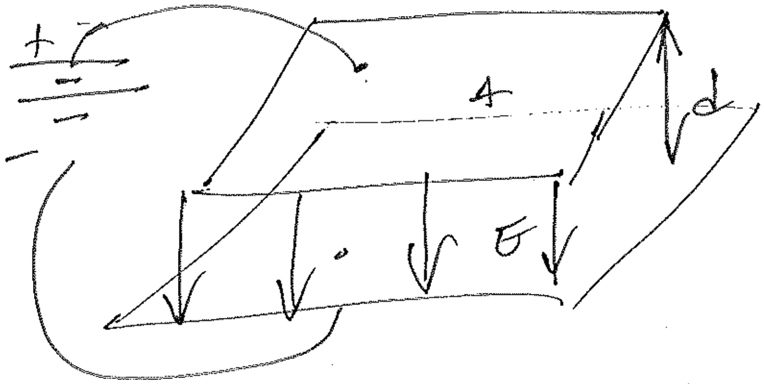
$W_E = 0 + \frac{1}{2}\int \epsilon \bar{E}^2 \, du \quad \frac{1}{2}\left(\int \frac{D^2}{\epsilon} \, du\right)$

No. 5505 Engineer's Computation Pad

Example

$$E = \frac{\rho}{2\epsilon_0}$$

$$\rho_s = \frac{dQ_s}{ds}$$



A parallel plate capacitor for which $C = \epsilon \frac{A}{d}$ has a $V = ct$ across the plates.
Find the stored energy in the electric field.

→ Fringing neglected (edge effects)

$$\boxed{\begin{aligned} dV &= -\bar{E}\cdot d\bar{l} \\ \bar{E} &= -\nabla V \end{aligned}}$$

$$\bar{E} = \left(\frac{V}{l}\right)\hat{a}_n \quad \text{between the plates}$$

$$= \left(\frac{V}{d}\right)\hat{a}_n \quad \text{and } E = 0 \text{ elsewhere.}$$

$$W_E = \frac{1}{2}\int_{volu} \epsilon E^2 \, dv = \frac{\epsilon}{2}\left(\frac{V}{d}\right)^2 \int_{volu} dv$$

$$= \frac{\epsilon}{2}\left(\frac{V}{d}\right)^2 A\cdot d$$

$$= \frac{\epsilon V^2 A}{2d} = \frac{1}{2}CV^2$$

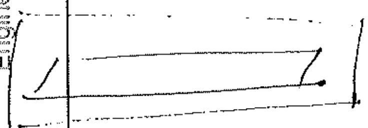
Second method

the total charge on one conductor maybe found from $\overline{D} = \epsilon \overline{E}$ at the surface via Gauss's law

$$\overline{D} = \epsilon \frac{V}{d} \hat{a}_n$$

$$\oint \overline{D} \cdot d\overline{S} = \int \rho \, dV = Q_{enc}$$

$$\oint \epsilon \overline{E} \cdot d\overline{S} = \int \rho \, dV = Q_{encl}$$



$$Q = |\overline{D}| \cdot A$$

$$= \epsilon \frac{V}{d} \cancel{\hat{a}_n} \cdot A$$

$$= \frac{\epsilon V A}{d}$$

$$W = \frac{1}{2} Q V = \frac{1}{2} \frac{\epsilon V A}{d} V = \frac{1}{2} \frac{\epsilon A V^2}{2} = \frac{1}{2} C V^2$$

Problem

Given the field $\bar{E} = \frac{K}{r}\hat{a}_r$ in cylindrical coords show that the work needed to move a point charge Q from any radial distance to a point at twice the radial distance is independent of R

$$dW = -Q\bar{E}\cdot d\bar{l} = -QE_r dr = -\frac{KQ}{r}dr$$

$$W = -KQ\int_{r_1}^{2r_1}\frac{dr}{r} = -K\ln 2$$

independent of r_1

Problem For a line charge $\rho_l = \frac{10^{-9}}{2}\frac{C}{m}$ on the z axis find V_{AB} when $A = (2m, \frac{\pi}{2}, 0)$ and $B = (4m, \pi, 5m)$

$$V_{AB} = -\int_B^A \bar{E}\cdot d\bar{l} \qquad \bar{E} = \frac{\rho_l}{2\pi\epsilon_0 r}\hat{a}_r$$

only radial $\bar{E}\cdot d\bar{l} \rightarrow E_r dr$

$$V_{AB} = -\int_B^A \frac{10^{-9}}{2(2\pi\epsilon_0 r)}dr = -9\ln r\Big|_4^2 = 6.24V$$

Problem Given a fiel

$$\bar{E} = -\left(\frac{16}{r^2}\right)\hat{a}_r \quad \frac{V}{m}$$ is spherical coords

find the potential of poin $(2m, \pi, \pi/2)$
with respect to $(4m, 0, \pi)$

Equipotential lines are concentric spherical shells $r = 2m$ surface A
$r = 4m$ surface B

$$V_{AB} = -\int_2^4 \frac{-16}{r^2}\, dr = -4V$$

Problem Find the potential at $r_A = 5m$
with respect to $r_B = 15m$ due to
a point charge $Q = 500\ pC$
at the origin and zero reference @ infinit

$$V_{AB} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

$$= \frac{500\times10^{-12}}{4\pi\left(10^{-9}/36\pi\right)}\left(\frac{1}{5} - \frac{1}{15}\right) = 0.6V$$

$\boxed{\text{the zero reference is not needed}}$

It is needed to find V_5 and V_{15}

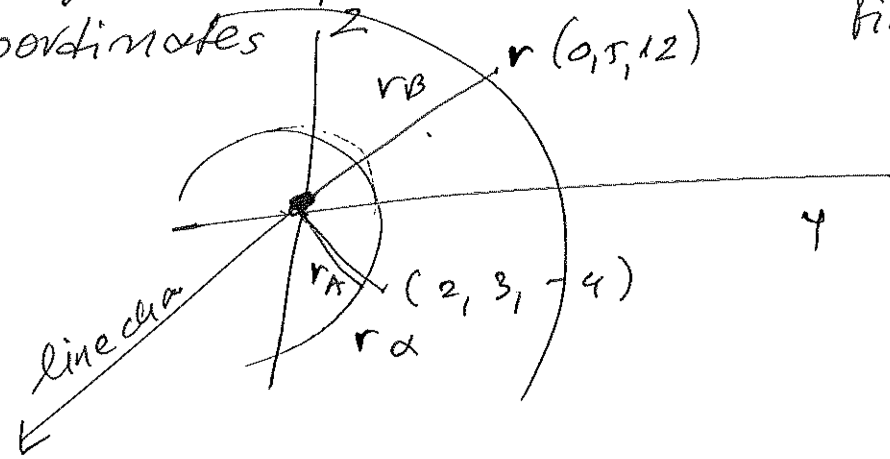
$$V_5 = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{5}\right) = 0.9\,V$$

$$V_{15} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{15}\right) = 0.3\,V$$

$$\boxed{V_{AB} = V_5 - V_{15} = 0.6\,V}$$

Problem

A line charge ρ_ℓ lies along the x-axis and the surface of 0 potential passes through the point $(0, 5, 12)$ m in cartesian coordinates. Find the potential at $(2, 3, -4)$



$r_A = \sqrt{9+16} = 5\text{m} \quad r_B = \sqrt{25+144} = 13\text{m}$

$$V_{AB} = -\int_{r_B}^{r_A} \frac{\rho_\ell}{2\pi\epsilon_0 r}\, dr = -\frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{r_A}{r_B}$$

$(= 6.88\,V)$

Problem

The electric field between two concentric cylintrical conductors at $r = 0.01$ m and $r = 0.05$ m is given by $\bar{E} = \left(\frac{10^5}{r}\right) \hat{a}_r \left(\frac{V}{m}\right)$ (fringinging neglected)

Find the energy stored in a 0.5 m length assume free space

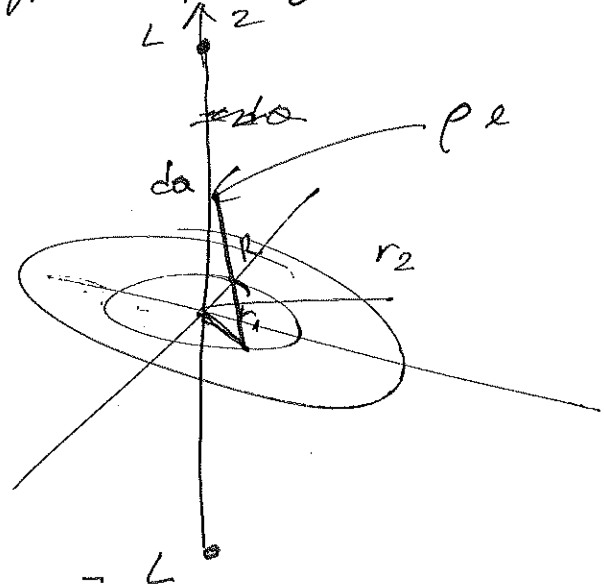
$$W_E = \frac{1}{2} \int \epsilon_0 E^2 dv =$$

$$= \frac{\epsilon_0}{2} \int_{h}^{h+0.5} \int_{0}^{2\pi} \int_{0.01}^{0.05} \left(\frac{10^5}{r}\right) r dr d\varphi dz$$

$$(= 0.224 \text{ J.})$$

Problem

Charge is distributed uniformly along a straight line of finite length 2L



Show that for two external points near the midpoint such that r_1 and r_2 are small compared to the length L the V_{12} is the same as for infinite line charge

$$\text{point } V_1 = \int_0^L \frac{\rho_e \, dz}{4\pi\epsilon_0 (z^2+r_1^2)^{1/2}}$$

$$= \frac{2\rho_e}{4\pi\epsilon_0}\left[\ln\left(z+\sqrt{z^2+r_1^2}\right)\right]_0^L$$

$$= \frac{\rho_e}{2\pi\epsilon_0}\left(\ln\left(L+\sqrt{L^2+r_1^2}\right)-\ln r_1\right)$$

$$V_2 = \frac{\rho_l}{2\pi\epsilon_0}\left[\ln\left(L+\sqrt{L^2+r_2^2}\right) - \ln r_2\right]$$

$$L >> r_1 \qquad L >> r_2$$

$$V_1 \approx \frac{\rho_l}{2\pi\epsilon_0}\left(\ln 2L - \ln r_1\right)$$

$$V_2 \approx \frac{\rho_l}{2\pi\epsilon_0}\left(\ln 2L - \ln r_2\right)$$

$$\boxed{V_{12} = V_1 - V_2 \approx \frac{\rho_l}{2\pi\epsilon_0}\ln\frac{r_2}{r_1}}$$

Problem

a spherical conducting shell of ratius α centered at the origin has a potential field

$$V = \begin{cases} V_0 & r \leq \alpha \\ V_0 \dfrac{\alpha}{r} & r > \alpha \end{cases}$$

(with 0 reference @ infinity). Find an expression for the stored energy

$$\bar{E} = -\nabla V = \begin{cases} 0 & r < \alpha \\ V_0 \dfrac{\alpha}{r^2} \hat{\alpha}_r & r > \alpha \end{cases}$$

$$W_E = \frac{1}{2} \int_{vol} \epsilon_0 E^2 \, dv = 0 + \frac{\epsilon_0}{2} \int_0^{2\pi} \int_0^{\pi} \int_{\alpha}^{\infty} \left(\frac{V_0 \alpha}{r^2} \right)^2 r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$= 2\pi \epsilon_0 V_0^2 \alpha$$

The total charge on the shell is?

$$Q = \varepsilon E A = \left(\frac{\epsilon_0 V_0 \alpha}{\alpha^2} \right) (4\pi \alpha^2) = 4\pi \epsilon_0 V_0 \alpha$$

$\oint \bar{D} \cdot d\bar{S}$

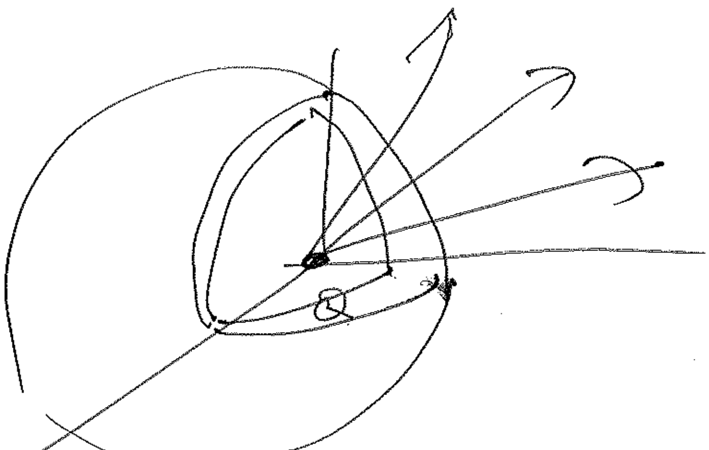
the potential @ the shell is $V = V_0 \Rightarrow W_E = \frac{1}{2} Q V$

energy stored in a spherical capacitor w/ other plate @ infinity

CONDUCTORS

Under static conditions the field outside a conductor is zero both tangential and normal components unless there exists a surface charge distribution

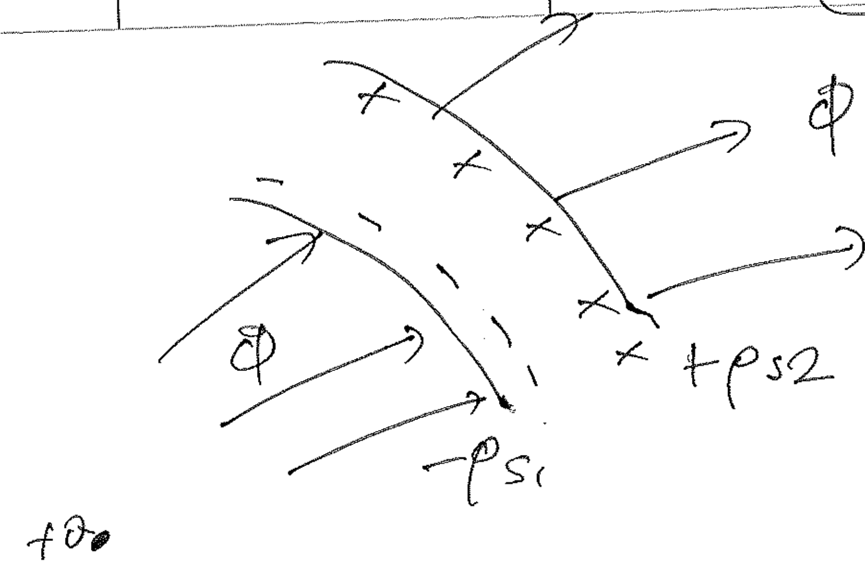
A surface charge does not imply a net charge in the conductor.



cosider a positive charge Q the origin of spherical coords. If the point charge is enclosed by an uncharged conducting ~~surface~~ spherical shell of finite thickness the field is

$$\overline{E} = \frac{+Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

except within the conductor wher $\overline{E} = 0$



The Coulomb forces caused by $+Q$ attract the conduction electrons to the inner surface where they create a ρ_{s1} of negative sign. Then the deficiency of e^- on the other surface constitutes a positive surface density ρ_{s2}. The flux terminates at $-\rho_{s1}$ there $E=0$ in between and outside there is more flux going out. Flux does not pass through the conductor and the net charge on the conductor remains zero.