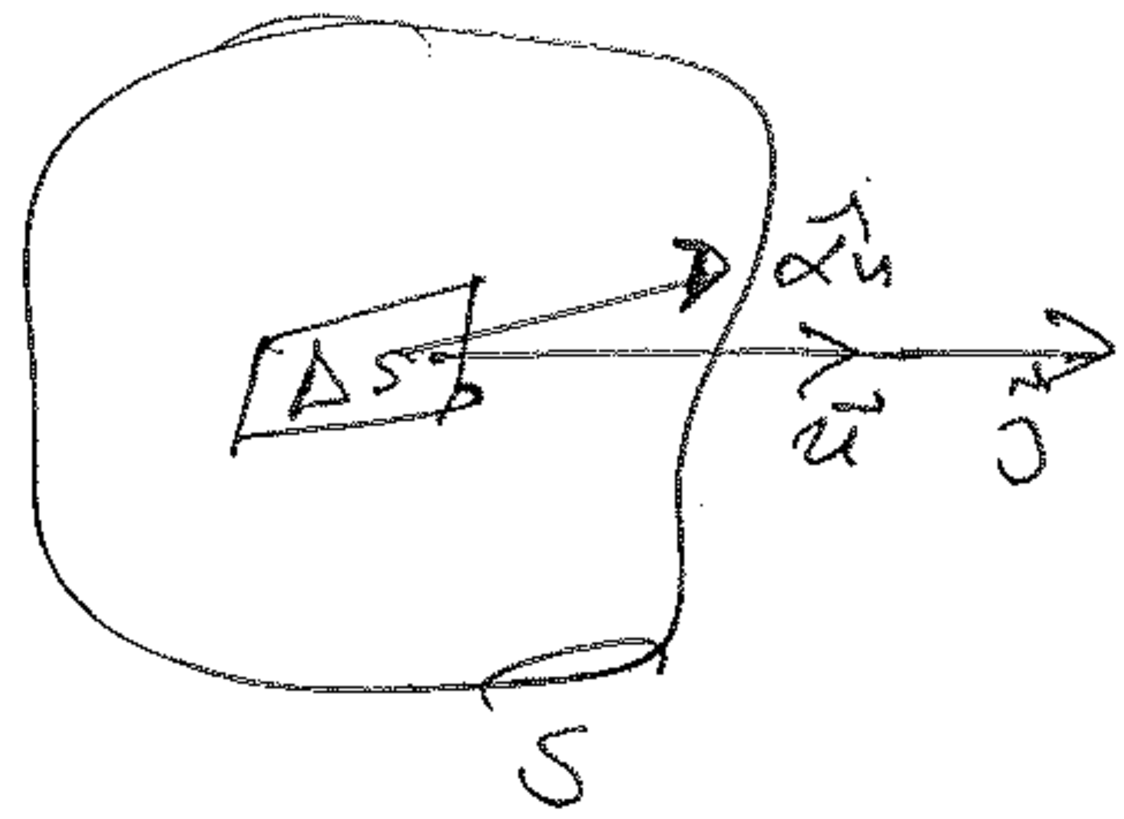
March 3

Steady Electric Currents

Steady motion of one kind of charge carriers, each of charge q (- for e) across an element of surface Δs with velocity $\vec{u}$



If N is the number of charge carriers per unit volume, then in time Δt each carrier moves a distance

$$\boxed{\vec{u}\,\Delta t}$$

the amount of charge passing through the surface Δs is

$$\Delta Q = N q \vec{u} \cdot \hat{a}_n \, \Delta s \, \Delta t \quad (c)$$

Current is the time rate change of charge

$$\Delta I = \frac{\Delta Q}{\Delta t} = N q \vec{u} \cdot \hat{a}_n \Delta s = N q \, \vec{u} \cdot \Delta \vec{s}$$

(A)

$\Delta \vec{S} = \hat{a}_n \Delta s.$

It is convenient to define a vector point function

$\boxed{\text{volume current density}}$

or

$\boxed{\text{current density}}$

$\bar{J}$ in Ampers per square meter

$$\bar{J} = N q \cdot \bar{u} \quad (A/m^2)$$

so that

$$\boxed{\Delta I = \vec{J} \cdot \Delta \bar{S}}$$

The total current I flowing through an arbitray surface S is then the flux of the vector $\bar{J}$ thru S

$$\boxed{I = \int_S \bar{J} \cdot d\bar{S} \quad (A)}$$

Note that $\underline{Nq}$ is charge per unit volume

we write $\boxed{\bar{J} = Nq\bar{u} = \rho \bar{u}}$ (A/m^2)

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relation between convection current density and the velocity of the charge carrier

In the case of <u>conduction currents</u> there may be more than one type of charges

electrons, holes, ions

drifting w/ different velocities

$$\bar{J} = Nq\bar{u} \rightarrow \text{generalize}$$

$$\bar{J} = \sum_i N_i q_i \bar{u}_i \qquad (A/m^2)$$

"Conduction" currents are the result of the drift motion of charge carriers under the influence of an applied electric field. The atoms remain neutral.

It can be justified analytically that for most conducting materials the average drift velocity is directly proportional to the $\bar{E}$ and that

$$\bar{J} = Nq\bar{u} \quad (A/m^2)$$

$$\boxed{\bar{J} = \sigma \bar{E}} \quad (A/m^2)$$

where the proportionality constant $\boxed{\sigma}$ is a macroscopic constitutive parameter of the medium called $\boxed{\text{conductivity}}$

Isotropic material for which the linear relation $\bar{J} = \sigma \bar{E}$ holds are called "Ohmic media"

Unit for $\boxed{\sigma}$ is A/V.m or siemens/m

Copper: the most commonly used conductor has a conductivity of 5.8×10^7 (S/m)

Rubber: has $\sigma_{rubber} = 10^{-15}$ S/m

Conductivity of materials varies over an extremely wide range

→ The reciprocal of conductivity is called resistivity in Ohm.meter

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We recall Ohm's law from circuit theory that V_{12} across a resistance R in which a current I flows from point 1 to point 2 is

$$\boxed{V_{12} = R \cdot I} \qquad \text{Ohm's law}$$

R is usually a piece of conducting material of length L

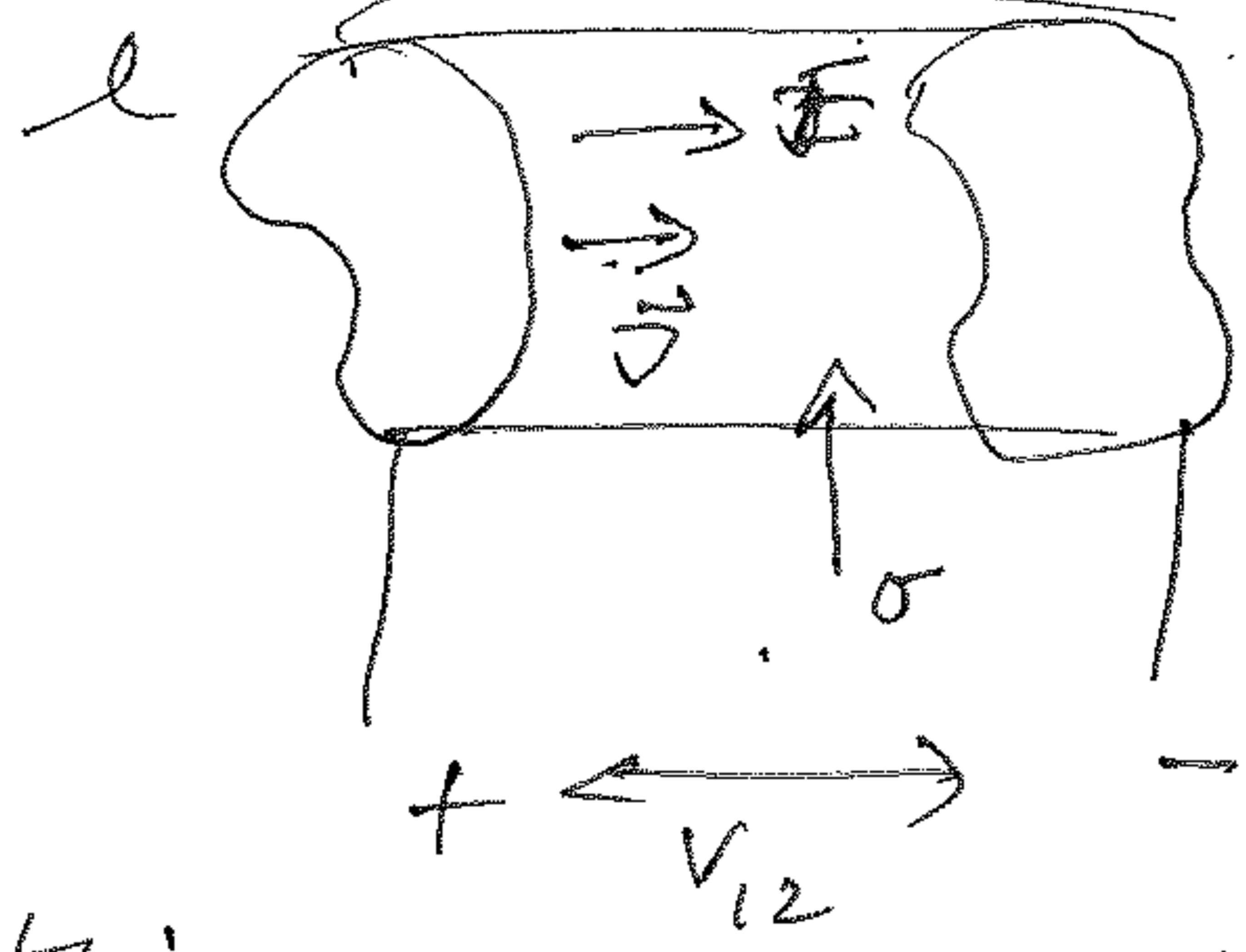
V_{12} is the voltage between terminals 1 and 2

I is the total current flowing from 1 to 2 through a finite cross section

$V_{12} = R \cdot I$ is not a point relation

$\bar{J} = \sigma \bar{E}$ is considered as the point form of Ohm's law:
it holds a all points in space and σ can be a function of space coordinates

Use the point form of Ohm's law to derive the voltage-current relationship of a piece of homogeneous material of conductivity σ, length l, and uniform cross section S



within the material

$$\bar{J} = \sigma \bar{E}$$

whe $\bar{J}$, $\bar{E}$ are in the direction of current flow

$$V_{12} = E\,l$$

$$E = \frac{V_{12}}{l}$$

$$I = \int \bar{J} \cdot d\bar{s} = JS$$

$$\boxed{\begin{aligned} J &= \frac{I}{S} \\ J &= \sigma \bar{E} \end{aligned}} \quad \begin{aligned} &= \frac{I}{S} \\ &= \sigma \frac{V_{12}}{l} \end{aligned} \quad \frac{I}{S} = \sigma \frac{V_{12}}{l}$$

$$\Rightarrow V_{12} = \left(\frac{l}{\sigma S}\right) I = R \cdot I \;!!!$$

This is the same as $V = R \cdot I$!!!
so we have a formula for the resistance of a straight piece of homogeneous material of a uniform ~~co~~ cross section for steady current.

$$R = \frac{l}{\sigma S} \quad (\Omega)$$

We could have started with

$$V_{12} = RI$$

as the experimental Ohm's law and applied it to a homogeneous conductor of length l and uniform cross section S.
Using $R = \dfrac{l}{\sigma S}$ we could go upwards and derive $\bar{J} = \sigma \bar{E}$ (A/m^2)

→ Determine the DC resistance of 1 Km of wire having 1 mm radius
(a) wire is made of copper
(b) aluminium

$$R = \frac{l}{\sigma S}$$

(a) copper $\sigma_{cu} = 5.8 \times 10^7 \ (S/m)$

$l = 10^3 \ (m)$

$S = \pi (10^{-3})^2 = \pi \, 10^{-6} \ m^2$

$$R_{cu} = \frac{l}{\sigma_{cu} S} = \frac{10^3}{5.80 \times 10^7 \times \pi \, 10^{-6}} = 5.49 \ \Omega$$

(b) $\sigma_{al} = 3.54 \times 10^7 \ S/m$

$$R_{al} = \frac{l}{\sigma_{al} \cdot S} = \frac{\sigma_{cu}}{\sigma_{al}} R_{cu} = \frac{5.80}{3.54} \times 5.49$$

$$= 8.99 \ \Omega$$

The reciprocal of resistance is called Conductance G $\quad G = \frac{1}{R} = \sigma \frac{S}{l} \ (S)$

Circuit Theory :

(1) When resistances R_1 and R_2 are connected in series (same current) the total resistance is

$$R_{series} = R_1 + R_2$$

(2) When resistances R_1 and R_2 are connected in parallel (same voltage)

$$\frac{1}{R_{\parallel}} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{or}$$

$$G_{\parallel} = G_1 + G_2$$

* EMF (electromotive force) and Kirchhoff's voltage law

$\boxed{\text{Static electric field is conservative}} \longrightarrow$

the line integral of the scalar static electric field intensity around any closed path is zero

$$\boxed{\oint_C \bar{E} \cdot d\bar{l} = 0}$$

For an ohmic material $\Longrightarrow$ $\bar{J} = \sigma \bar{E}$

$$\boxed{\oint_C \frac{1}{\sigma} \bar{J} \cdot d\bar{l} = 0}$$

A steady current cannot be maintained in the same direction in a close circuit by an electrostatic field

A steady current in a circuit is the result of motion of charge carriers which collide w/ atoms and dissipate energy in the circuit. The energy must come from a non-conservative field since a charge carrier completing a closed circuit "in a conservative field neither gains nor loses energy.. The source can be e.g. batteries!"

electric generators, thermocouples photovoltaics, etc.
These when connected to a circuit provide the driving force for the charge carriers.

When a resistor is connected between terminals 1 and 2 of the battery completing the circuit the total electric field intensity is the electrostatic $\bar{E}$ by charge cumulation and the "impressed" E_i cause by chemical action in the battery so

$$\bar{J} = \sigma(\bar{E} + \bar{E}_i)$$

$\bar{E}_i$ exists only in the battery

$\bar{E}$ is non 0 everywhere

$\bar{E} + \bar{E}_i = \frac{\bar{J}}{\sigma}$ take integral

$$\boxed{\oint_C (\bar{E}+\bar{E}_i)\cdot d\bar{l} = \oint_C \frac{1}{\sigma}\,\bar{J}\cdot d\bar{l}}$$

with source ~~of~~ $\bar{E}_i$

$$\boxed{\oint_C \frac{1}{\sigma}\,\bar{J}\cdot d\bar{l} = 0}$$

with no source

If the resistor has conductivity σ, length l and uniform cross-section S then

$$J = \frac{I}{S} \quad \text{and}$$

$$\oint_C \frac{1}{\sigma}\,\bar{J}\cdot d\bar{l} = RI = \text{emf}$$

(ideal emf source has 0 resistance)

In general for many emfs and many resistors

$$\sum_j V_j = \sum_k R_k I_k \quad (v)$$

Kirchhof's law

$$\sum V = \sum RI$$

It states that around a closed path in an elecrtic circuit the algebraic sum of the emf's (voltage rise) is equal to the algebraic sum of the voltage drops across the resistances

It applies to any closed path in a network

The direction of tracing the path can be arbitralily assigned and the currents in the different resistances need not be the same

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** Continuity and Current law

The principle of conservation of charge is one of the fundamental postulates. Electric charges may not be created or destroyed.

Consider an arbitrary volume V bounded by surface S. A net charge Q exists in the region. If a net current I flows across the surface OUT of the region the charge must decrease at a rate that equals the current.

Conversely if a net charge flows INTO the region the charge must increase at a rate equal to the current. The current leaving the region is the total outward flux of the current density vector thru the surface S

$$I = \oint_S \bar{J} \cdot d\bar{S} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_{volume} \rho \, dv$$

divergence theorem again:

surface integral of $\bar{J}$ → volume integral of $\bar{\nabla}\cdot\bar{J}$

$$\boxed{\int_{volume} \bar{\nabla}\cdot\bar{J}\, dv = -\int_{volume} \frac{\partial \rho}{\partial t}\, dv}$$

I moved the time derivative of ρ inside the volume integral and made it partial in case ρ is a function of space-coordinates as well. The equation above holds regardless of the volume

$$\boxed{\bar{\nabla}\cdot\bar{J} = -\frac{\partial \rho}{\partial t}} \quad A/m^3$$

point relation → equation of continuity → result of charge conservation

For steady currents $\frac{\partial \rho}{\partial t} = 0$

the charge density does not vary w/ time

$$\boxed{\bar{\nabla} \cdot \bar{J} = 0}$$

Steady electric currents are divergence-less (or solenoidal)

this hold also whe $\rho = 0$ (no flow source)

It mean that field lines or streamlines of steady currents close upon themselves unlike the electrostatic ones of the $\bar{E}$ <u>that originate and end</u> on charges.

$$\bar{\nabla} \cdot \bar{J} = 0 \Rightarrow \oint_S \bar{J} \cdot d\bar{S} = 0$$

$$\Rightarrow \sum_j I_j = 0 \quad (A)$$

Kirchhof's current law: The algebraic sum of all currents flowing out of a junction in an electric circuit is zero.

* We said in conductors $\rho = 0$ $\bar{E} = 0$
lets prove this and calculate the time

$$\left.\begin{aligned}\bar{J} &= \sigma \bar{E} \quad (A/m^2) \\ \bar{\nabla}\cdot\bar{J} &= -\frac{\partial \rho}{\partial t}\end{aligned}\right\}$$

$$\sigma\, \bar{\nabla}\cdot\bar{E} = -\frac{\partial \rho}{\partial t}$$

In a simple medium $\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\epsilon}$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0}$$

$$\boxed{\rho = \rho_0 e^{-(\sigma/\epsilon)t}} \quad (C/m^3)$$

ρ_0 = intial charge density at $t=0$
ρ, ρ_0 can be functions of space coordinates
The charge density at a given location will decrease with time exponentially
An intial charge density ρ_0 will decay to $1/e$ or 36.8% of its value in a time

$$\boxed{\tau = \frac{\epsilon}{\sigma}} \quad (s)$$

We call τ : relaxation time

For copper ~~in air~~

$\sigma = 5.8\ 10^{7}\ S/m$

$\epsilon \approx \epsilon_0 = 8.85 \times 10^{-12}\ \frac{F}{m}$

$\tau = \frac{\epsilon}{\sigma} = 1.52 \times 10^{-19}\ s.$

Very fast. Superluminal fast: what is going on

$\boxed{\rho = 0}\ \boxed{\bar{E} = 0}$

For insulator τ can be hours or days

Assert | Boundary conditions for current density

Governing equations for steady current density $\bar{J}$ in the absence of non conservative energy sources

$$\bar{\nabla} \cdot \bar{J} = 0 \qquad \oint_S \bar{J} \cdot d\bar{S} = 0$$

$$\bar{\nabla} \times \left(\frac{\bar{J}}{\sigma} \right) = 0 \qquad \oint_C \frac{1}{\sigma} \bar{J} \cdot d\bar{l} = 0$$

$$\boxed{\bar{\nabla} \times \bar{E} = 0}$$

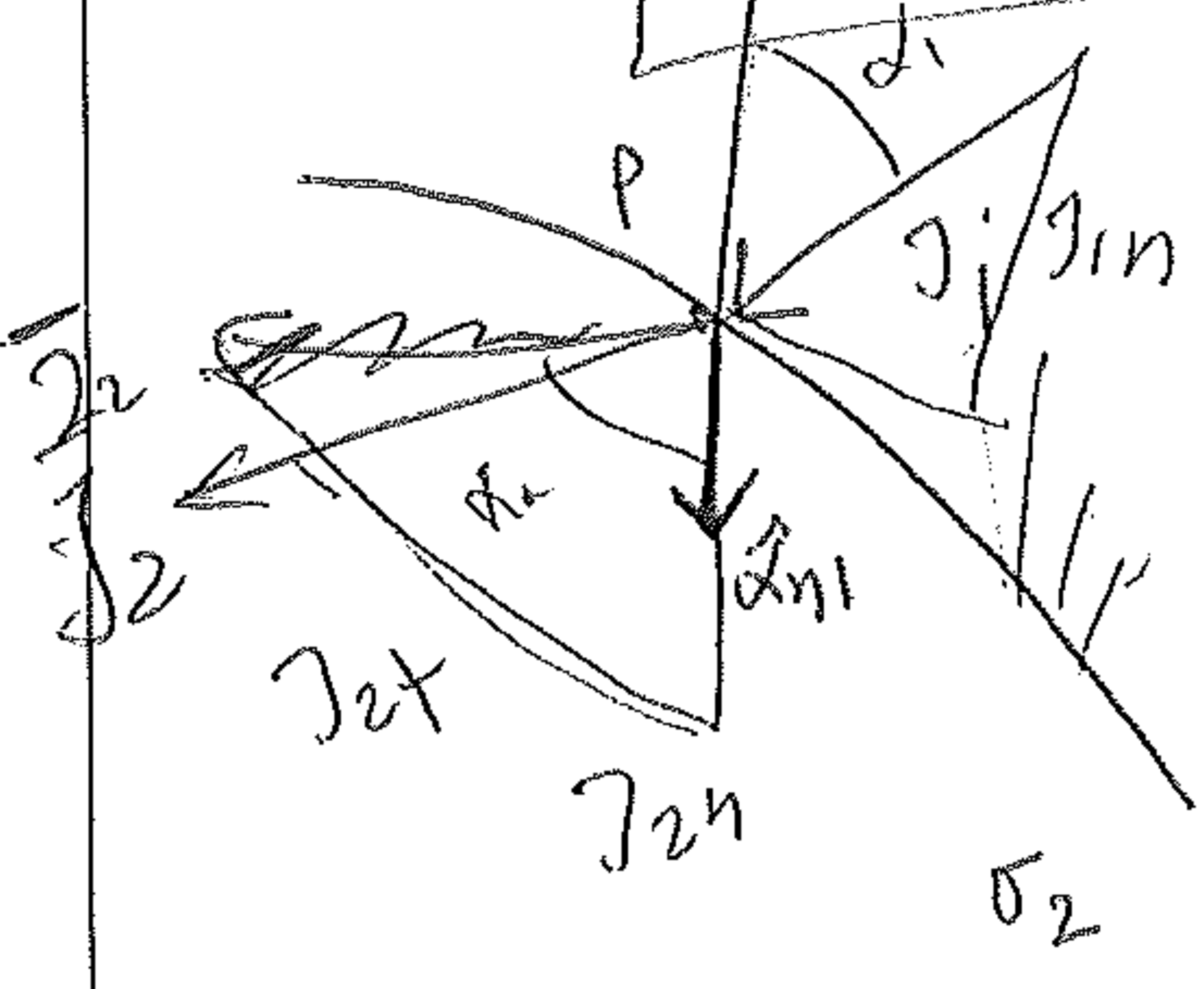
A the interface of two ohmic media with conductivities σ_1, σ_2 boundary conditions for normal and tangenial component of $\bar{J}$

$$\boxed{J_{1n} = J_{2n}}$$

the normal component of a divergenceless vector field is continuous

tangenial component of a curl-free vector is continuous across an interface

$$\boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}}$$



reference unit normal outward for medium 2

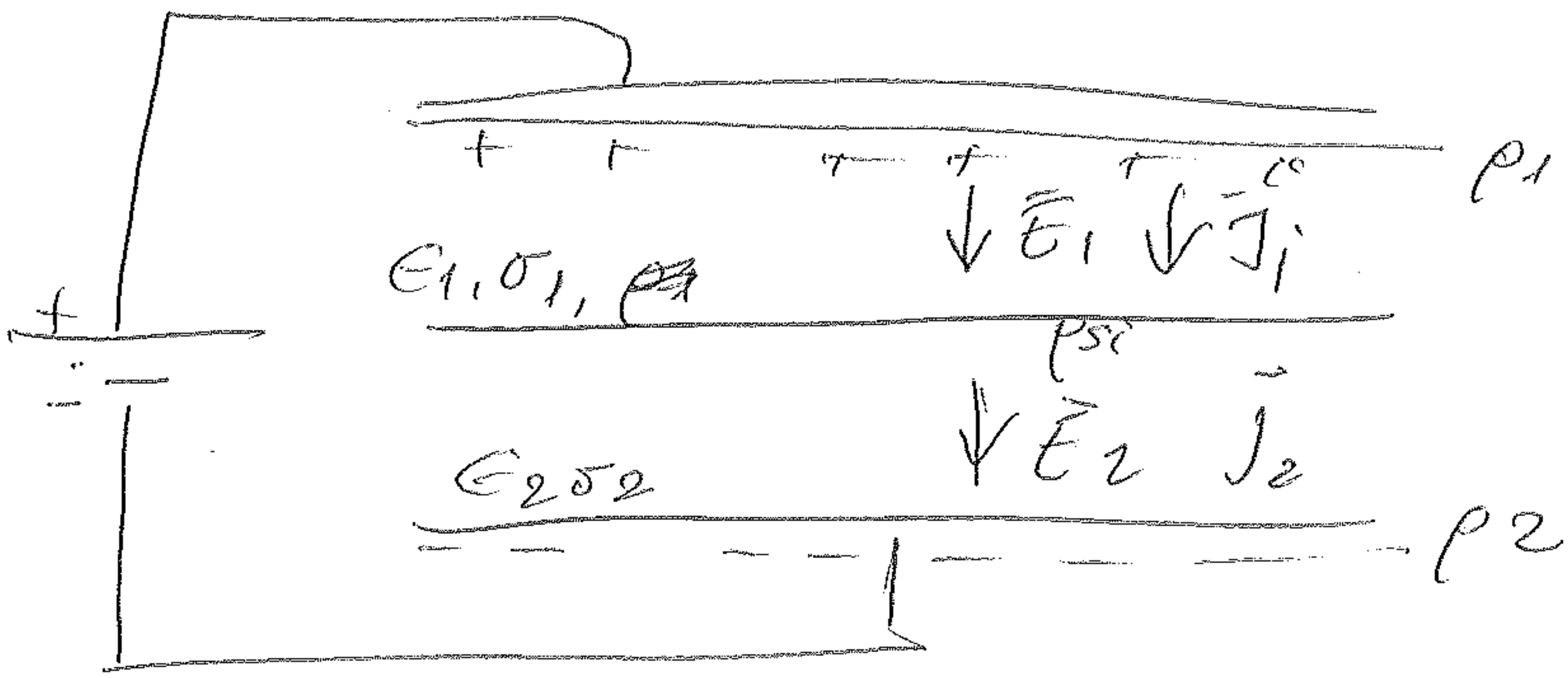
March 4

2 An emf V is applied across a parallel-plate capacitor of area S. The space between the conductive plates is filled w. different "lossy" dielectrics of ~~thickness~~ d_1 and d_2 (lossy means with finite σ) ~~and~~ permittivities ϵ_1, ϵ_2 and conductivities σ_1 and σ_2

(a) current density between plates

(b) Electric field intensities in both dielectrics

(c) surface charge densities on the plates + interface



(a) Continuity of J normal component

$$\boxed{\begin{aligned} J_{1n} &= J_{2n} \rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n} \\ \epsilon_1 E_{1n} - \epsilon_2 E_{2n} &= \rho_{s\,interface} \end{aligned}}$$

the current in both media is the same

Kirchhoff's law voltag

$$V = (R_1+R_2)\,I = \left(\frac{d_1}{\sigma_1 S} + \frac{d_2}{\sigma_2 S}\right) I$$

$$J = \frac{I}{S} = \frac{V}{(d_1/\sigma_1) + (d_2/\sigma_2)} = \frac{\sigma_1 \sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}$$

(A/m^2)

(b) neglect fringing

$$V = E_1 d_1 + E_2 d_2$$
$$\sigma_1 E_1 = \sigma_2 E_2 \quad (J_1 = J_2)$$

$$E_1 = \frac{\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2} \quad V/m$$

$$E_2 = \frac{\sigma_1 V}{\sigma_2 d_1 + \sigma_1 d_2} \quad V/m$$

(c) $$\rho_{s1} = \epsilon_1 E_1 = \frac{\epsilon_1 \sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2} \quad C/m^2$$

$$\rho_{s2} = -\epsilon_2 E_2 = \frac{\epsilon_2 \sigma_1 V}{\sigma_2 d_1 + \sigma_1 d_2} \quad C/m^2$$

The negative sign : $\bar{E}_2$ and the outward normal out the lower plate are 1l

parenthesis

$\sigma_1 E_{1n} = \sigma_2 E_{2n}$

$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$ (interface)

$$\rho_s \text{ interface} = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right) E_{2n}$$

$$= \left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2}\right) E_{1n}$$

If one medium (say 2 is better conductor)

$\sigma_2 >> \sigma_1 \quad \frac{\sigma_1}{\sigma_2} \to 0 \quad \rho_s = \epsilon_1 E_{1n} = (\rho_{1n})$

$$\rho_{si} = \left(\epsilon_2 \frac{\sigma_2}{\sigma_1} - \epsilon_1\right)\left(\frac{\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}\right) E_{1n}$$

$$= \frac{(\epsilon_2 \sigma_1 - \epsilon_1 \sigma_2) V}{\sigma_2 d_1 + \sigma_1 d_2} \quad C/m^2$$

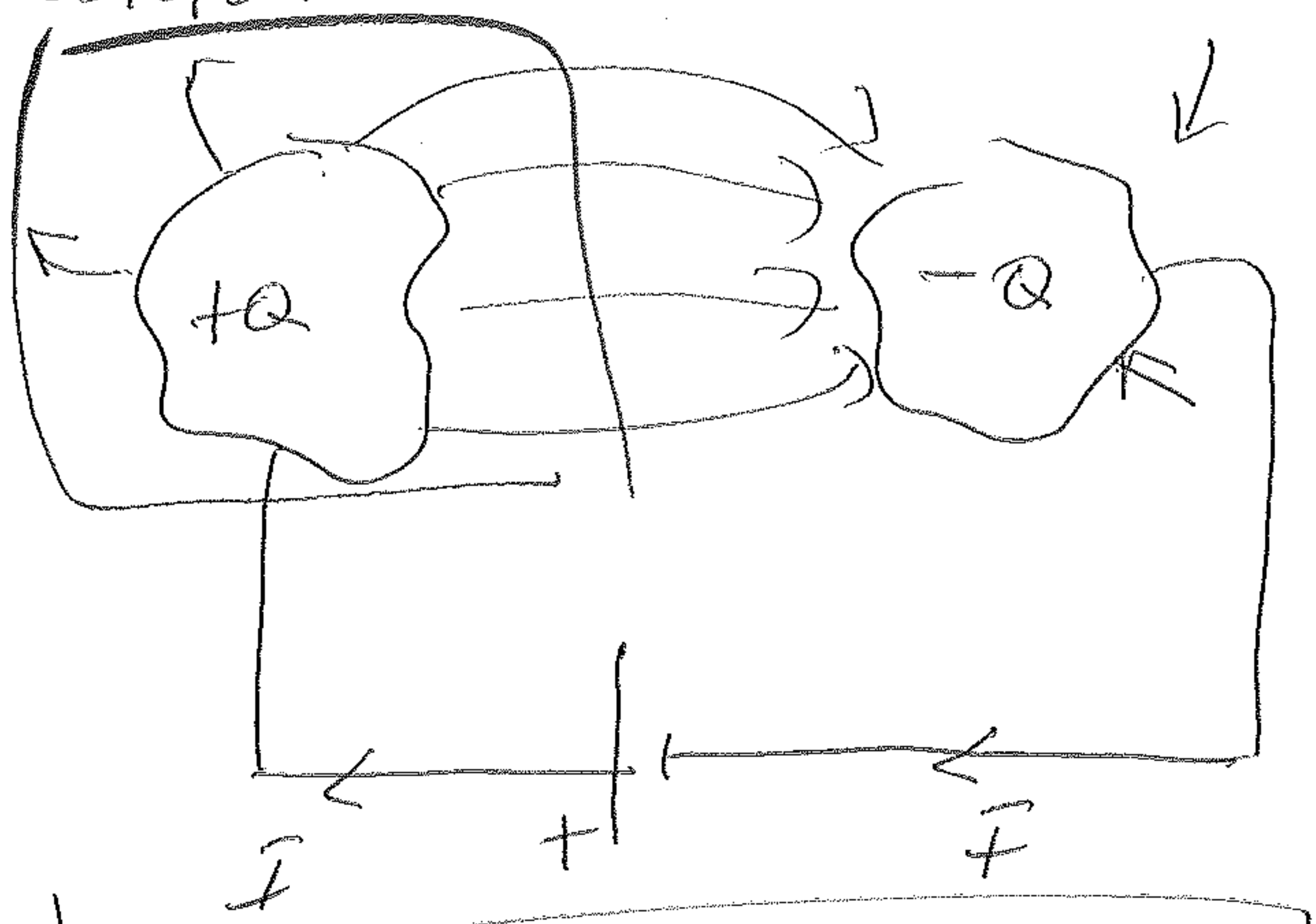
$\boxed{\rho_{s2} \neq -\rho_{s1} \quad \text{but} \quad \rho_{s1} + \rho_{s2} + \rho_{si} = 0}$

situation w/ both static charges and steady current $\Rightarrow$

static E
steady B } electromagnetic field.

$\boxed{\bar{J} = \sigma \bar{E}}$

Resistaunce



$$\boxed{C = \frac{Q}{V} = \frac{\oint_S \epsilon\, \bar{E} \cdot d\bar{S}}{-\int_L \bar{E} \cdot d\bar{l}}}$$

→ surface enclosing positive conductor

→ line from lower to higher potential

In a lossy dielecric (small but non-0 conductivity) current flows from + to − and a current density is established in the medium

$\bar{J} = \sigma \bar{E}$ ensures that in an isotopic medium the "lines" will be the same

$$R = \frac{V}{I} = \frac{-\int_L \bar{E} \cdot d\bar{l}}{\oint_S \bar{J} \cdot d\bar{S}} = \frac{-\int_L \bar{E}\, d\bar{l}}{\oint \sigma \bar{E} \cdot d\bar{S}}$$

$$\boxed{RC = \frac{C}{G} = \frac{\epsilon}{\sigma}}$$

If ϵ and σ of the medium have the same space dependence of ,
If the medium is homogenius, ~~the~~
if the capacitance is known the resistance can be found fro $\boxed{\epsilon/\sigma}$

e.g. Coaxial cable :

$$C_1 = \frac{2\pi\epsilon}{\ln\left(\frac{b}{a}\right)}$$

$$R_1 = \frac{\epsilon}{\sigma}\frac{1}{C_1} = \frac{\epsilon}{\sigma 2\pi\epsilon}\ln\left(\frac{b}{a}\right)$$

(Ωm)

Procedure for computing resistance of a piece of conducting material

1. Choose appropriat coordinate system for given geometry

2. assume potential difference V_0 between conductor terminals

3. find $\bar{E}$ within the conductor (If the material is homogeneous having a constant conductivity solve Laplacian $\bar{\nabla}^2 V = 0$ for V

   and get $\bar{E} = -\bar{\nabla} V$

4. find total current

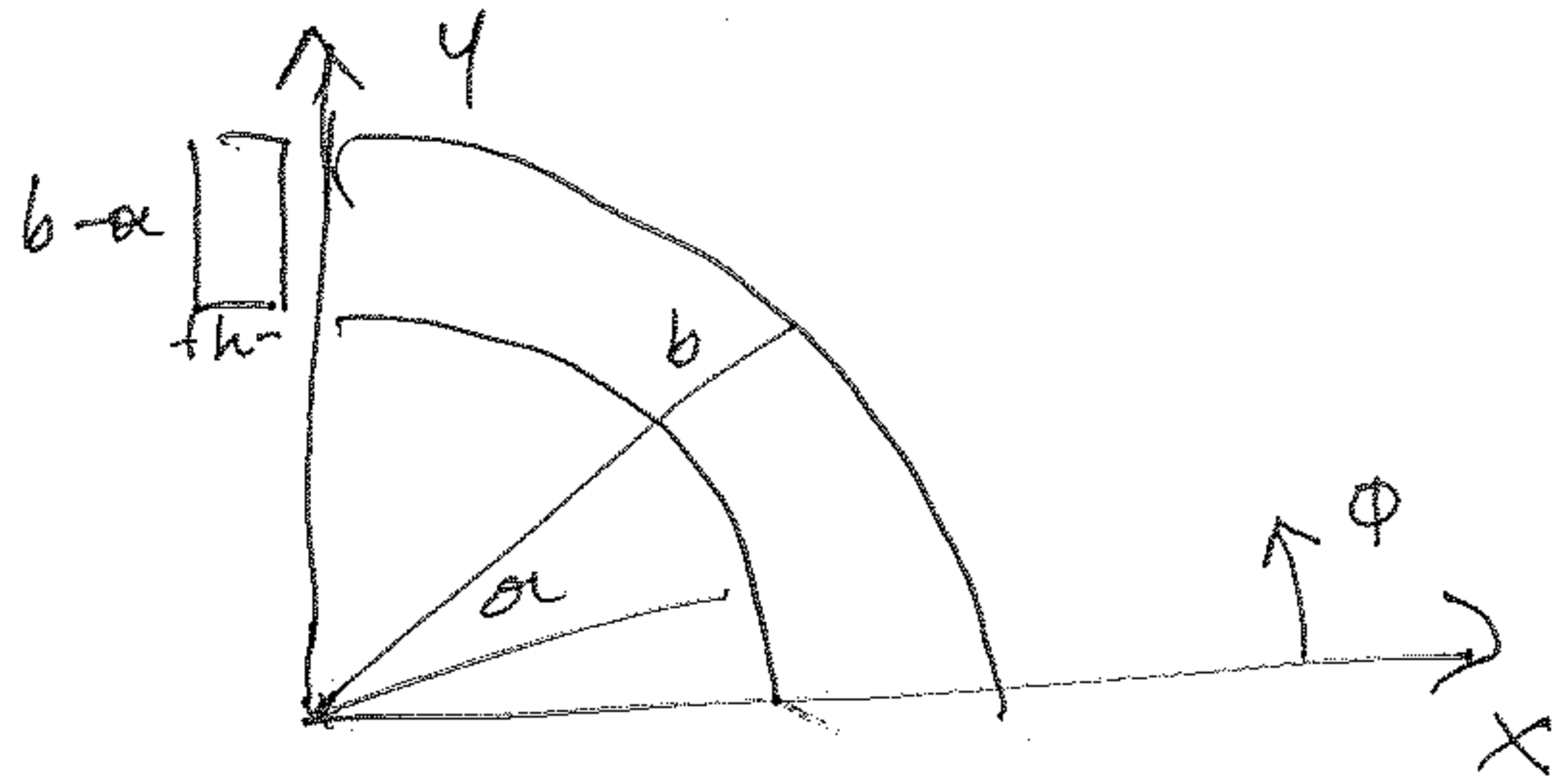
$$\boxed{I = \int_S \bar{J} \cdot d\bar{S} = \int_S \sigma \bar{E} \cdot d\bar{S}}$$

   wher S is the cross sectional area over which I flows

5 find $R = \frac{V_0}{I}$

A conducting material of uniform thickness h and conductivity σ has the shape of a quarter of a flat circular washer with inner radius a and outer b

Determine the resistance



$$\left(\frac{d^2V}{d\varphi^2} = 0 , \quad \Rightarrow \quad V = \right.$$ (boundary conditions)

$$\bar{J} = \sigma \bar{E} = -\sigma \bar{\nabla} V$$

$$I = \int_S \bar{I} . d\bar{s}$$

$$R = \frac{V_0}{I} = \frac{\pi}{2\sigma h \ln(b/a)}$$

next page first

1) cylintrical

2) assume V_0, say $V=0$ on the end face at $y=0$ and $V=V_0$ at $x=0$

$V=0$ @ $\varphi=0$
$V=V_0$ @ $\varphi=\pi/2$

$V=V(\varphi)$ $\boxed{\dfrac{d^2V}{d\varphi^2}=0}$ Laplace

$V=C_1\varphi+C_2$ (boundary condition)

$$V=\frac{2V_0}{\pi}\varphi$$

$$\bar{J}=\sigma\bar{E}=-\sigma\bar{\nabla}V$$

$$=-\hat{a}_\varphi\,\sigma\frac{\partial V}{r\,\partial\varphi}=-\hat{a}_\varphi\frac{2\sigma V_0}{\pi r}$$

integrate $\bar{J}$ over the $\varphi=\pi/2$ surface at which $d\bar{s}=-\hat{a}_\varphi\,h\,dr$

$$I=\int_S\bar{J}\cdot d\bar{s}=\frac{2\sigma V_0}{\pi}h\int_a^b\frac{dr}{r}$$

$$=\frac{2\sigma h V_0}{\pi}\ln\frac{b}{a}$$

$$R = \frac{V_0}{I} = \frac{n}{2\sigma h \ln\left(\frac{b}{a}\right)}$$

Note, it is not obvious how $\bar{J}$ varies with r for a given I.

Without $\bar{J}$, $\bar{E}$ and V_0 cannot be determined.

* Power dissipation

electric field → conduction electron drift → collide w/ atoms on lattice sites → energy is transmitted from the electric fields to the atoms in thermal vibrations

The work ΔW done by an $\bar{E}$ field moving a charge q a distance Δl is

$dW = q\bar{E}\cdot(\Delta\bar{l})$ ~~th~~ this corresponds to a power

$$\boxed{P = \frac{dW}{dt} = q\,\bar{E}\cdot\bar{u}}$$

$\vec{u}$ is the drift velocity

total power delivered to all charge carriers in a volume dv

$$dP = \sum_i p_i = \bar{E}\cdot\left(\sum_i N q\,\bar{u}_i\right) d\,\mathrm{vol}$$

$$dP = \bar{E}\cdot\bar{J}\ d\,\mathrm{volume}$$

$$\frac{dP}{dv} = \bar{E}\cdot\bar{J} \quad (W/m^3)$$

The point function $\bar{E}\cdot\bar{J}$ is a ~~po~~ power density under steady current conditions. for a given volume V the total electric power converted into heat is

$$P = \int_{volume} \underline{\bar{E} \cdot \bar{J}\, dv} \quad \text{(Watt)}$$

(not Joule)

This is Joule's law

In a conductor of a constant cross section

$dv = \underline{ds}\,\underline{dl}$ with dl measured $\parallel \bar{J}$

$$\boxed{P = \underbrace{\int_L E\, dl}_{V} \underbrace{\int_S J\, ds}_{I} = VI}$$

$V = RI$

$$= I^2 R \quad (W)$$

ohmic power representing heat dissipated in resistance R per unit time

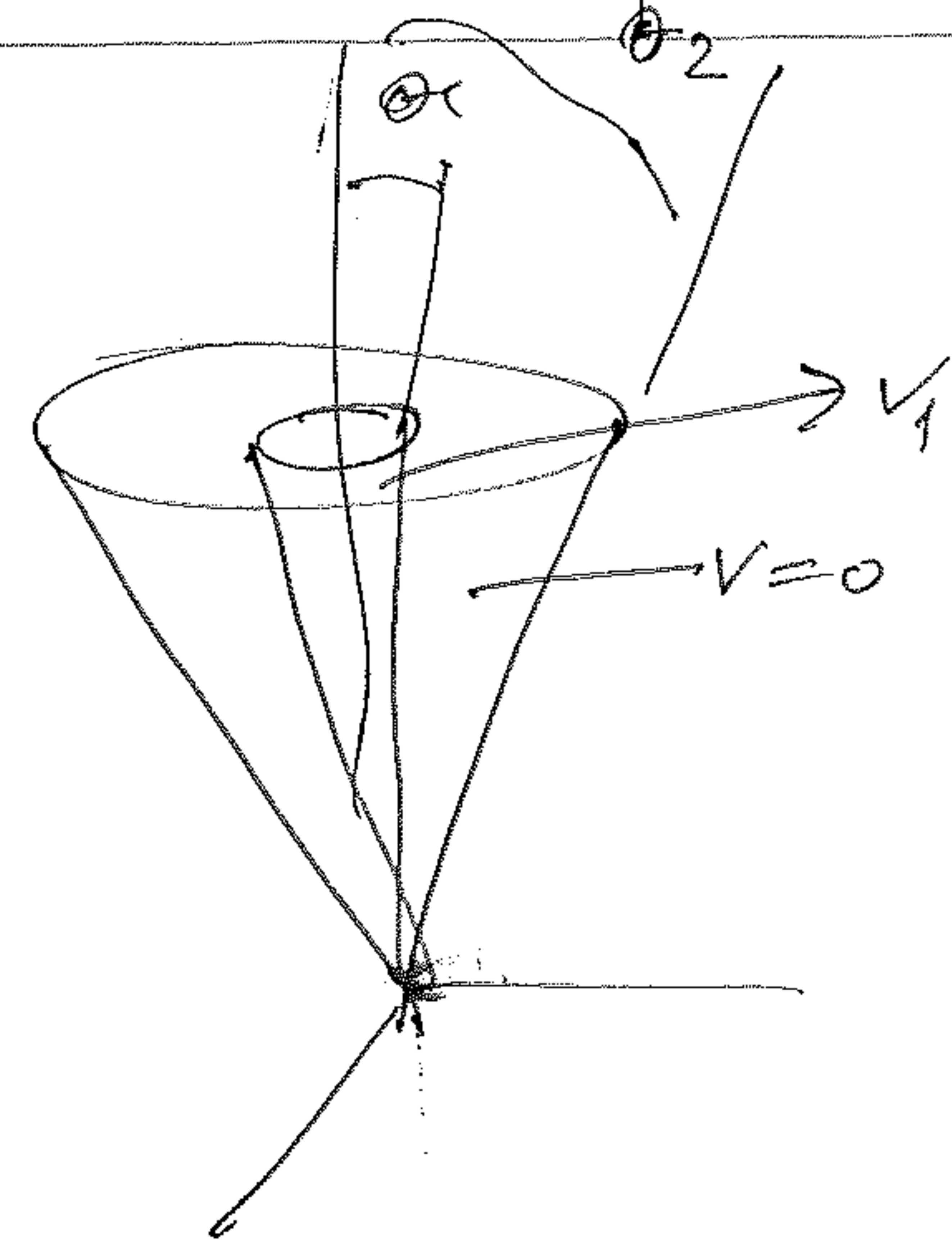
* Find the capacitance of an isolated <u>spherical shell</u> of radius a

The potential of such a conductor with a zero reference at infinity is

$$V = \frac{Q}{4\pi\epsilon_0 a}$$

$$C = \frac{Q}{V} = 4\pi\epsilon_0 a$$

* ~~Find the capacitance between 2 spherical shells of radius a separated by a distance $d \gg a$~~



Solve Laplace equation for the region between 2-coaxial cones as shown in Figur
inside $V_1 = V_1$ and outsid $V = 0$
cone vertices are insulated. The potential is constant with r and φ

$$\frac{1}{r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0$$

$$\sin\theta \frac{dV}{d\theta} = A$$

$$V = A \ln\left(\tan\frac{\theta}{2}\right) + B$$

$$V_1 = A \ln\left(\tan\frac{\theta_1}{2}\right) + B \qquad 0 = A \ln\left(\tan\frac{\theta_2}{2}\right) + B$$

$$V = V_1 \frac{\ln\left(\tan\frac{\theta}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}{\ln\left(\tan\frac{\theta_1}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}$$