Feb 16/17 → [redacted]

$$\operatorname{div} \bar{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$$

Divergence of a vector field (scalar)

$\operatorname{div} \bar{E} = \frac{\rho}{\epsilon}$ Gauss' law differential

del operator $\bar{\nabla} \equiv \frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z$

$\bar{\nabla} \cdot \bar{A} \equiv \operatorname{div} \bar{A}$ $\quad \bar{\nabla} \cdot \bar{E} = \operatorname{div} \bar{E} = \frac{\rho}{\epsilon}$

$$\oint_{surf} \epsilon \bar{E} \cdot d\bar{S} = \cancel{\oint \epsilon E} \int_{v} \rho \, dv = Q_{enclosed}$$

$\rho = \bar{\nabla} \cdot \epsilon \bar{E}$

Green's theorem in 3-d (divergence theorem)

$$\oint_{surface} \epsilon \bar{E} \cdot d\bar{S} = \int_{volume} (\bar{\nabla} \cdot \epsilon \bar{E}) \, dv$$

$$dV_{potential} = -\bar{E} \cdot d\bar{l} = \bar{\nabla} V \cdot d\bar{r}$$

$$\Rightarrow \boxed{\bar{E} = -\bar{\nabla} V} \quad V = -\int \bar{E} \cdot d\bar{l}$$

$$\left(V_2 - V_1 = -\int_1^2 \bar{E} \cdot d\bar{l}\right)$$

gradient of a scalar field = vector field

The gradient of a potential ~~fie~~ ϕ function is a vector field that is everywhere normal to equipotential surfaces

PART A

* CONDUCTORS IN STATIC ELECTRIC FIELD

So far we mostly talked about electric field of stationary charge distribution in free space or air.

What about material?

→ conductors
→ semiconductors
→ insulators (dielectrics)

In some crude atomic model

the electrons in outermost shells of the atoms of <u>conductors</u> are loosely held and migrate from one atom to the other

Most metals are like that

In dielectrics the outer electrons are held together; they cannot be "liberated" even when we put them in fields

Semi-conductors are in-between (band theory of solids, filled, semi-filled bands and transitions from band to band)

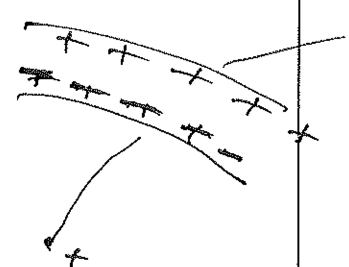
The macroscopic electrical property of a material medium is characterized by a ~~to~~ parameter called <u>conductivity</u>. We don't care about this yet because we talk about static electric fields in materials and not about current flow.

We talk about the electric field and charge distribution inside the bulk and on the surface of a conductor

Assume some +, or − charge inside the conductor. The charges will move reach the conductor surface and

1

Inside a conductor under static conditions
$\rho = 0$ $\bar{E} = 0$

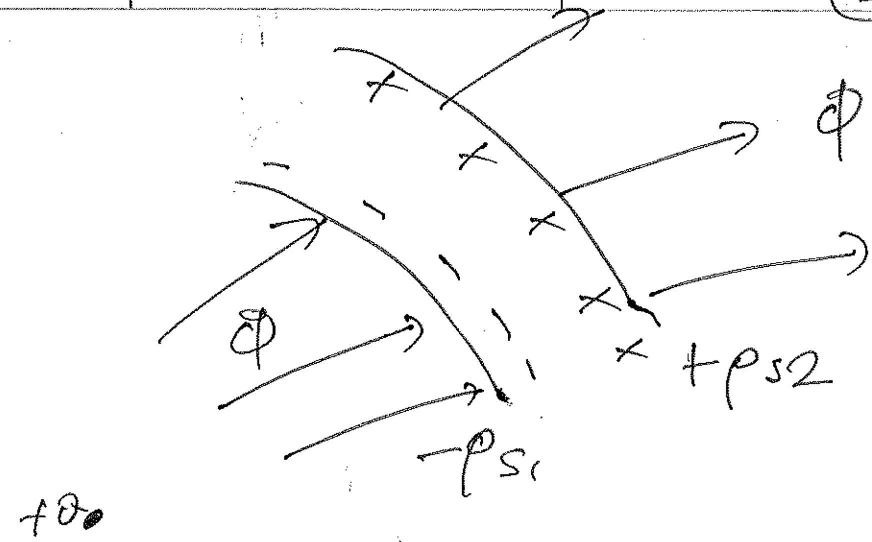


Under static conditions the $\bar{E}$ field on a conductor surface is everywhere normal to the surface $\Rightarrow$ i.e

he surface of a conductor is an equipotential surface under static conditions

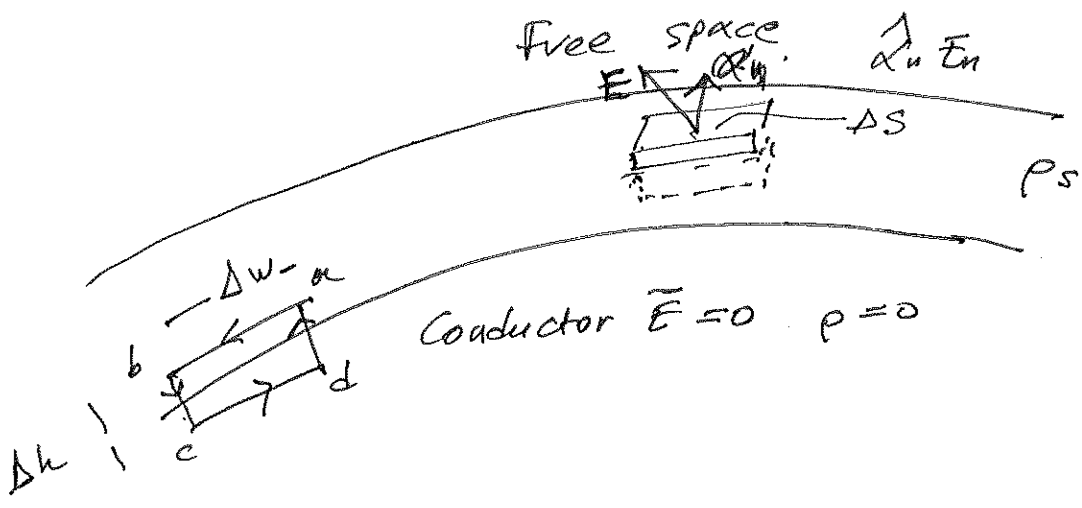
Since $\bar{E} = 0$ everywhere inside the conductor $\bar{E} = -\bar{\nabla} V \Rightarrow V = \text{constant}$

the potential is the electrostatic same everywhere



The Coulomb forces caused by $+Q$ attract the conduction electrons to the inner surface where they create a ρ_{s1} of negative sign. Then the deficiency of e^- on the other surface constitutes a positive surface density ρ_{s2}. The flux terminates at $-\rho_{s1}$ there $E=0$ in between and outside there is more flux going out. Flux does not pass through the conductor and the net charge on the conductor remains zero.

Interface between conductor and free-space



~~Assume some positive or negative charge is introduced at the interior of a conductor~~

$$\oint_{\substack{\text{surface}\\ abcda}} \bar{E}\cdot d\bar{l} = \underset{(\alpha_b)}{E_{\parallel}\,\underset{\Delta h \to 0}{\Delta W}} = 0 \quad \Rightarrow \quad (\bar{E} = 0)$$

$\Rightarrow$ $\boxed{E_{\parallel} \text{ (tangential component to surface)} = 0}$

Tangential component of the $\bar{E}$ field on a conductor surface is zero.

What about the normal? $\rightarrow$ take Gaussian surface $\rightarrow$ thin pillbox

top face of the pillbox in free space
bottom face in the conductor where $\bar{E} = 0$

Gauss' law

$$\oint_{surf} \mathbf{E}\cdot d\bar{S} = E_n \cancel{\Delta S} = \frac{\rho_s}{\epsilon_0} \cancel{\Delta S}$$

$$E_n = \frac{\rho_s}{\epsilon_0}$$

The normal component of the $\vec{E}$ field at a conductor - free space interface (boundary) is equal to the surface charge density on the conductor divide by the permitivity of free space

Basic

Boundary conditions Conductor - free space interface
$E_{\parallel} = 0$ $E_{\perp} = \dfrac{\rho_s}{\epsilon_0}$ (2)

When an uncharged conductor is placed in a static electric field the external field will cause the electrons ~~to move~~ inside the conductor to move at one direction and the net positive charges at another direction These INDUCED charges will distribute on the conductor surface and

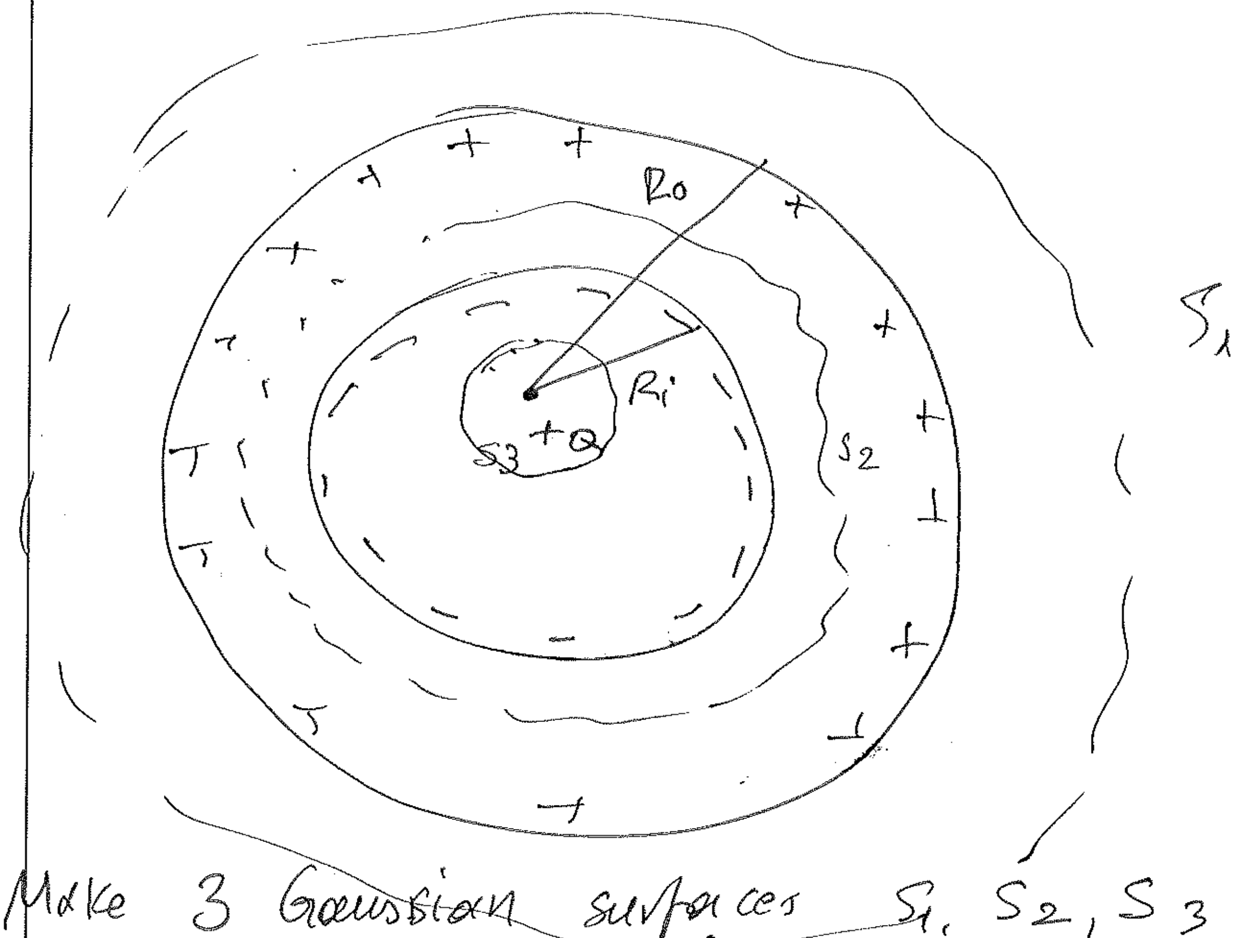
Create an induced field in such a way that they cancell the external field both inside the conductor and tangenially on the surface

When the surface charge distribution is in equilibrium (1) + (2) hold.

EXAMPLE

A positive point charge Q is at the center of a spherical conducting shell of inner radius R_i and outer radius R_o

Determine $\bar{E}$ and V(potential) as functions of radial distance R



Make 3 Gaussian surfaces S_1, S_2, S_3

Spherical symmetry → use Gauss's law for $\bar{E}$ and integrate to find V

$E_R = E_r \cdot \hat{a}_R$ everywhere

Three regions

1. $R > R_o$

2. $R_i \leq R \leq R_o$

3. $R < R_i$

a) $R > R_o$ (Gaussian surface 1)

$$\oint_{surf} \bar{E} \cdot d\bar{s} = E_r 4\pi R^2 = \frac{Q}{\epsilon_0}$$

$$E_r^{(1)} = \frac{Q}{4\pi\epsilon_0 R^2}$$

The field is the same as that of point charge Q without the presence of the conductor. The potential referring to the point @ infinity is

$$V_1 = -\int_{-\infty}^{R} E_{R_1} dR = \frac{Q}{4\pi\epsilon_0 R}$$

b) $R_i \le R \le R_o$ (G-surface S_2)

$E_{R2} = 0$ and the conducting shell is an equipotential body

$$V_2 = V_1 \Big|_{R=R_o} = \frac{Q}{4\pi\epsilon_0 R_o}$$

c) $R < R_i$ (G-surface S_3)

$$E_{R3} = \frac{Q}{4\pi\epsilon_0 R^2}$$

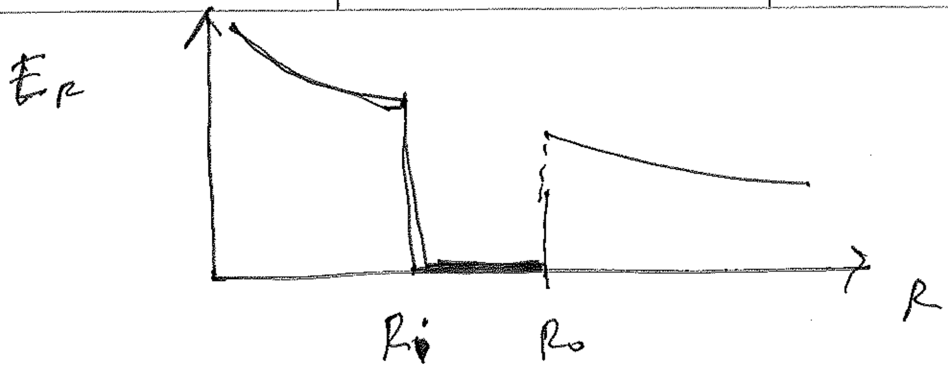
$$V_3 = -\int E_{R3}\, dR + C$$

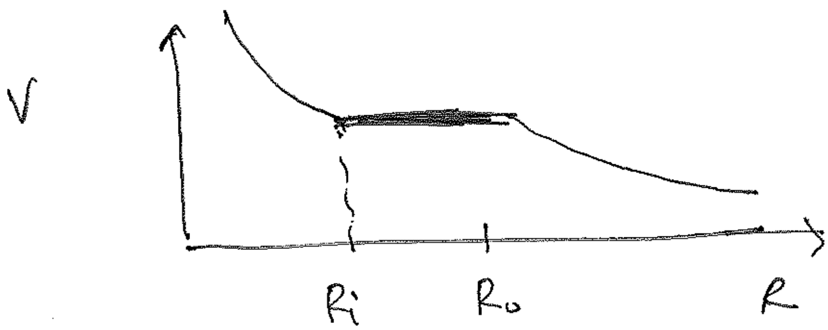
$$= \frac{Q}{4\pi\epsilon_0 R} + C$$

the integration constant C is determined by requiring V_3 @ $R = R_i$ to be equal to $V_2 = \frac{Q}{4\pi\epsilon_0 R_o}$

$$C = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_o} - \frac{1}{R_i}\right) \Rightarrow$$

$$V_3 = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R} + \frac{1}{R_o} - \frac{1}{R_i}\right)$$


E_R
R_i
R_o
R


V
R_i
R_o
R

$$\left.\begin{aligned} \bar{\nabla}\cdot\epsilon\bar{E} &= \rho \\ \bar{E} &= -\bar{\nabla} V \end{aligned}\right\}.$$

$$\bar{\nabla}\cdot\left(\epsilon\bar{\nabla}V\right) = -\rho$$ (ϵ can be a function of position)

For a simple medium, homogeneous ϵ is constant

$$\boxed{\bar{\nabla}^2 V = -\frac{\rho}{\epsilon}} \longrightarrow \text{Poisson's equation}$$

new operator $\bar{\nabla}^2$ the Laplacian operator (divergence of gradient) or

$$\bar{\nabla}\cdot\bar{\nabla}$$

it states that the Laplacian of V equals $-\frac{\rho}{\epsilon}$ for a simple medium whe ϵ is the permittivity and ρ is the volume charge density (which maybe a function of space coordinates)

$$\bar{\nabla}^2 V = \bar{\nabla}\cdot\bar{\nabla} V =$$

$$\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\cdot$$

$$\left(\frac{\partial}{\partial x}V_x\hat{a}_x + \frac{\partial}{\partial y}V_y\hat{a}_y + \frac{\partial}{\partial z}V_z\hat{a}_z\right)$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon}\left(\frac{V}{m^2}\right)$$

$$\bar{\nabla}^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\bar{\nabla}^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)$$

$$+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}$$

At points in a simple medium where there is no free charge $\rho = 0$ and

$$\boxed{\bar{\nabla}^2 V = 0}$$ Laplace's equation

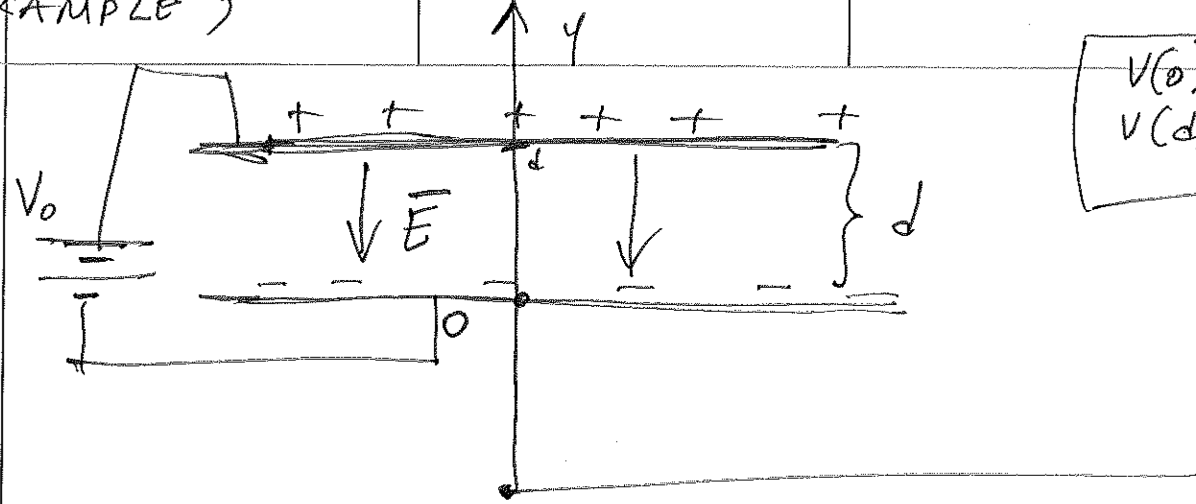
This is the governing equation for problems involving a set of conductors such as capacitors maintained at diverent potentials. Once V is found $\bar{E}$ can be ~~determined for~~ from

$$-\bar{\nabla} V$$

and charge distribution on the conductor surfaces can be determined from

$$\rho_s = \epsilon E_\perp$$

(EXAMPLE)



$V(0) = 0$
$V(d) = V_0$

Two plates of a parallel plate capacitor are separated by distance d and maintained at potential 0 and V_0 as shown in figure. Assuming no fringing effects at the edges

(a) determine the potential at any point between the plates

(b) the surface charge densities at the plates

* no fringing means that the $\bar{E}$ distribution is the same as if infinetely large and there is no variation of V in x, y directions

$$\frac{d^2}{dy^2} V = 0$$

(I use $\frac{d^2}{dy^2}$ bc I have no other space coordinates)

Integrate wrt y

$$\frac{dV}{dy} = C_1$$ constant to be determined

integrate again

$$V = C_1 y + C_2$$

Boundary conditions —

At $y=0$ $V=0$ $\Rightarrow$

$y=d$ $V=V_0$

$$C_1 = \frac{V_0}{d} \qquad C_2 = 0$$

$$\Rightarrow \quad V = \frac{V_0}{d} y$$

the potential increases linearly from $y=0$ to $y=d$

(b) In order to find surface charge densities must find $\bar{E}$ at the conducting plates at $y=0$ and $y=d$

$$\bar{E} = -\frac{dV}{dy}\hat{a}_y = -\frac{V_0}{d}\hat{a}_y$$ constant and independent of y

the direction of $\bar{E}$ is opposite to the direction of increasing V

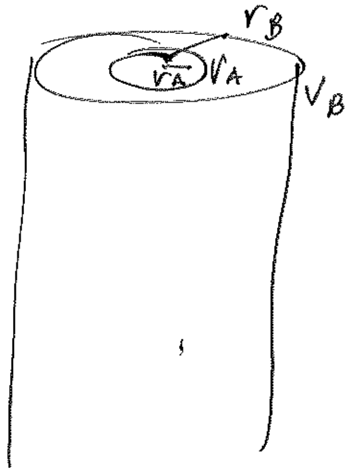
The surface charge densities at the conducting plates are obtained by using $E_\perp = E_n = \bar{E}\cdot\hat{a}_n = \frac{\rho}{\epsilon}$

lower plate $\hat{a}_n = \hat{a}_y$ $\quad E_{nl} = -\frac{V_0}{d}$ $\quad \rho_{sl} = -\frac{\epsilon V_0}{d}$

upper plate $\hat{a}_n = -\hat{a}_y$ $\quad E_{nu} = \frac{V_0}{d}$ $\quad \rho_{su} = \frac{\epsilon V_0}{d}$

Electric field lines in electrostatics field begin from positive charges and end in negative charges

Solve
w/ Gauss' law



find the potential function for the region between two concentric right circular cylinders where $V_A =$ $V_B =$

Potential is constant with φ and z

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

$\cancel{V_A = C_1 \ln r_1 + C_2}$

$\cancel{V_B = C_1 \ln r_2 + C_2}$

Integrate

$$r\frac{dV}{dr} = C$$

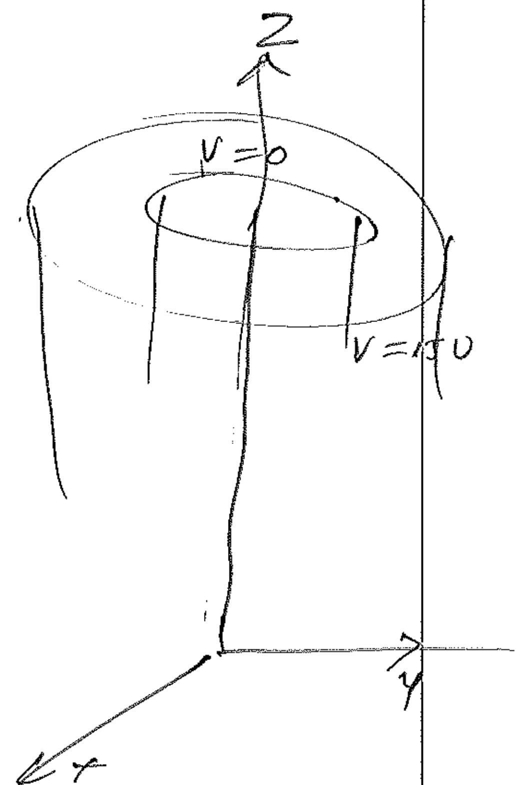
$$\frac{dV}{dr} = \frac{C}{r} \qquad V = C \ln\frac{r}{r_A} + V_A$$

$$\cancel{V = C_1 \ln\frac{r}{r_A}} \qquad C = \frac{V_B - V_A}{\ln\frac{r_B}{r_A}}$$

$$V = \frac{(V_B - V_A)}{\ln\frac{r_B}{r_A}} \ln\frac{r}{r_A} + V_A$$

Find the potential function and $\bar{E}$ for the region between 2 concentric right circular cylinder $V=0$ @ $r=1m$
$V=150\,V$ @ $r=20m$

Potential constant with φ, z



$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)=0$$

$$r\frac{dV}{dr}=A$$

$$V=A\ln r+B$$

$$0=A\ln 0.001+B$$
$$150=A\ln 0.020+B$$

$$A=50.1$$
$$B=345$$

$$V=50\ln r+345 \quad (V)$$

$$\bar{E}=\frac{50}{r}(-\hat{a}_r)\ \frac{V}{m}$$

FREE CHARGES NEAR CONDUCTING BOUNDARIES

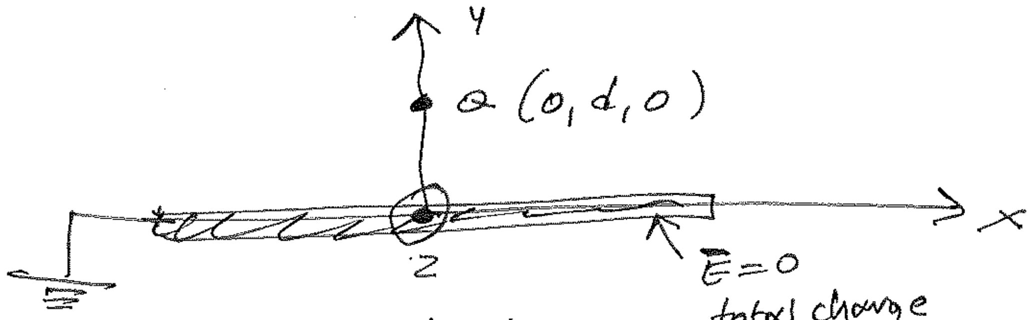
METHOD OF IMAGES

Class of electrostatic problems with boundary conditions that appear to be difficult to satisfy if Laplace's equation is to be solved directly, but the conditions on the boundaries/surfaces in these problems ~~can be determined~~ can be set up by appropriate "image" (equivalent) charges and the potential distributions can be straightforward to determine

Replace bounding surfaces by appropriate equivalent charge is called "method of images"

P4

Physical arrangement



total charge −Q with som ρs

grounded plane conductor

ground = 0 potential

positive charge located @ distance d from conducting plane (y > 0)

The formal procedure: solve Laplace in Cartesian coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

which holds for y > 0 except where the Q is

The solution V(x, y, z) should satisfy the following conditions

1.) $V(x, 0, z) = 0$ @ all points on the grounded plane the potential is ϕ

2.) @ points close to Q the potential approaches that of the charge alone

$$V \longrightarrow \frac{Q}{4\pi\epsilon_0 R}$$

5) At points far from Q ($x \to \pm\infty$, $y \to \pm\infty$, $z \to \pm\infty$)

the potential approaches ϕ

9) The potential function is even wrt x, y coordinates

$$V(x, y, z) = V(-x, y, z)$$
$$V(x, y, z) = V(x, y, -z)$$

Try to ~~solve~~ construct a solution for V that satisfies all 1–4 is a bit tifficult

Let's reason: the presence of Q at $y=d$. induces $-Q$ at the plate ~~resulting in surface density ρ_s~~

remove the plate and replace it with the $-Q$ at $y=-d$ then the potential

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^+} - \frac{1}{R^-}\right) \Bigg\} \boxed{\text{in the } y>0 \text{ region}}$$

$$R_+ = (x^2 + (y-d)^2 + z^2)^{1/2}$$
$$R_- = (x^2 + (y+d)^2 + z^2)^{1/2}$$

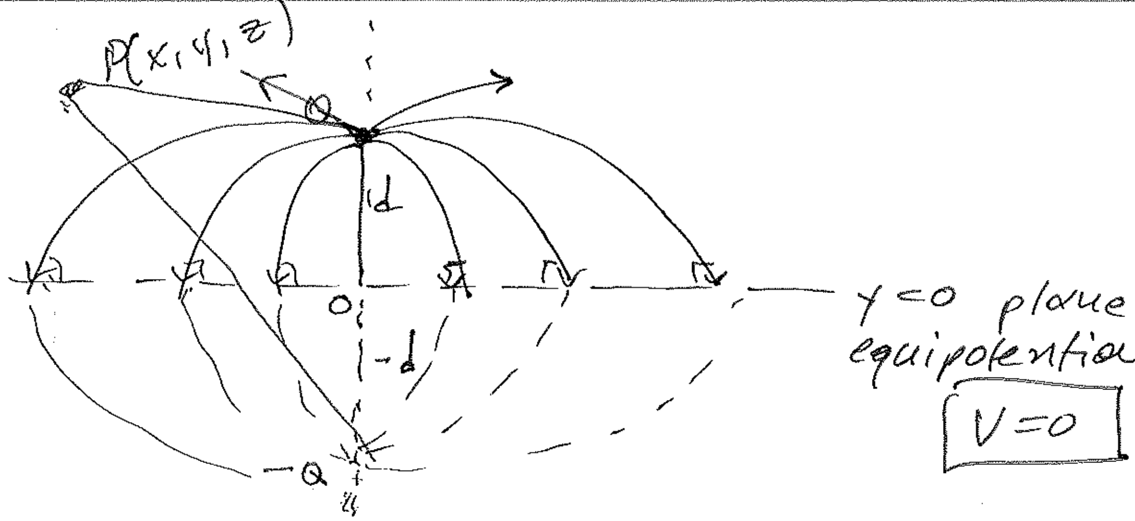


Image charge.

a solution of Laplace's equation that satisfies the given boundary conditions is a unique solution. (uniqueness theorem).

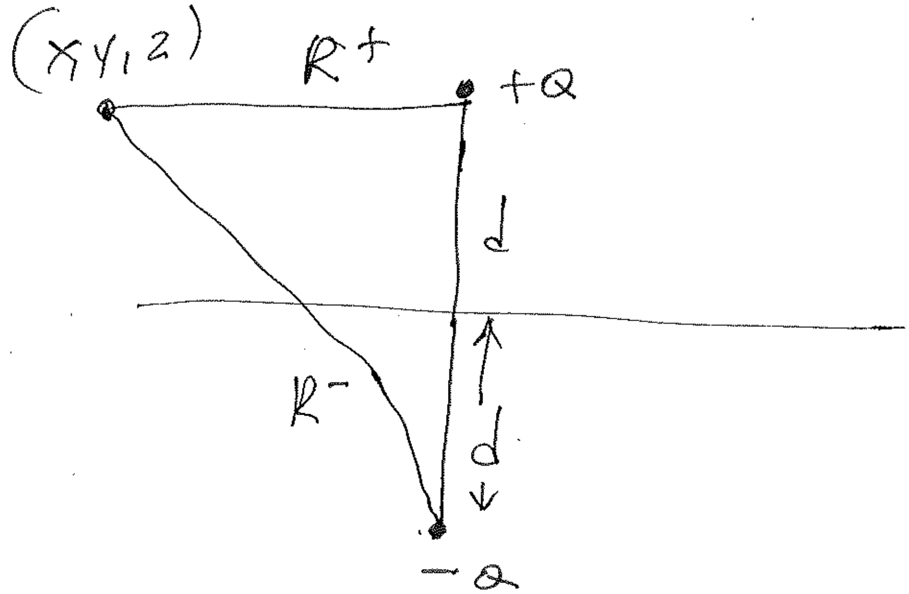
$\boxed{\bar{E} \text{ can be found for } y > 0 \text{ form}}$

$$\bar{E} = -\bar{\nabla} V$$

It is the same as the field between 2 point charges $+Q$, $-Q$ spaced at distance $2d$

$$\cancel{E_y(y=0)} = -\bar{\nabla}\left(\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^+} - \frac{1}{R^-}\right)\right).$$ solve w/ cylindrical coord.

Since the surface of the conducting plane represents an interface joining 2 solutions of Laplace's equations, namely $V=0$ the discontinuity in the electric field is accomodated by a surface charge density $\rho_s = -\frac{qd}{2\pi(d^2+r^2)^{3/2}}$



$r^2 = x^2 + z^2$

$\rho_s = \frac{dQ}{ds} \Rightarrow$

$Q = \int \rho_s \, dS$ = cylindrical) [redacted]

= [redacted].