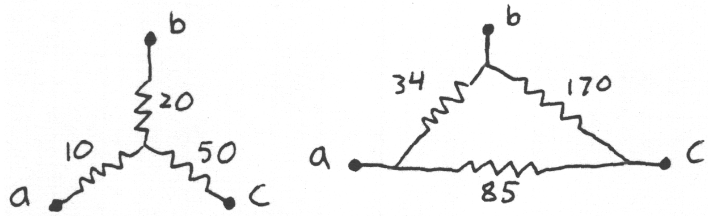
1 *Purcell 4.20 A black box with three terminals a, b, and c contains nothing by three resistors and connecting wire. Measuring the resistance between pairs of terminals, we find* $R_{ab} = 30$ *ohm,* $R_{ac} = 60$ *ohm, and* $R_{bc} = 70$ *ohm. Show that the contents cound be either of the following.*



Is there any other possibility? Are the two boxes completely equivalent, or is there an external measurement that would distinguish between them?

For the first box, the resistance between any two terminals involves two of the resistors in series with the third resistor extraneous. For example, $R_{ab} = 10\ ohm + 20\ ohm = 30\ ohm$.

For the second box, the resistance between any two terminals involves one resistor in parallel with the other two in series. For example,

$$R_{ab} = \left(\frac{1}{34\ ohm} + \frac{1}{85\ ohm + 170\ ohm}\right)^{-1} = 30\ ohm.$$

The other two are easily verified.

These are the only two ways to make these three resistances with only three resistors.

For the two arrangements to be electrically identical, they must both draw the same currents given the same input voltages V_a, V_b, and V_c. The details are somewhat messy. I'll leave it to you to verify that given the input voltages V_a, V_b, and V_c, both arrangements draw the currents

$$I_a = \frac{1}{170}(7V_a - 5V_b - 2V_c),$$

$$I_b = \frac{1}{170}(6V_b - V_c - 5V_a),$$

$$I_c = \frac{1}{170}(3V_c - V_b - 2V_a).$$

Note that $I_a + I_b + I_c = 0$ as it must.

2 *Purcell 4.25 A charged capacitor C discharges through a resistor R. Show that the total energy dissipated in the resistor agrees with the energy originally stored in the capacitor. Suppose someone objects that the capacitor is never really discharged because Q only becomes zero for* $t = \infty$*. How would you counter this objection?*

Assume that the capacitor initially has charge Q on it. The current as a function of time is

$$i(t) = i_o e^{-t/\tau}$$

where $\tau = RC$ and $i_o = V_o/R = Q/CR$. The power dissipated in a resistor is $P = i^2R$, so the total energy dissipated is

$$E = \int_0^\infty P\,dt = \int_0^\infty R\,i_o^2 \exp\left(-2t/\tau\right) dt = R\,i_o^2\frac{\tau}{2} = R\frac{Q^2}{C^2R^2}\frac{RC}{2} = \frac{1}{2C}Q^2.$$

This was the initial energy stored in the capacitor.

We can find the time it takes for the charge left on the capacitor to be one electron. The charge as a function of time is

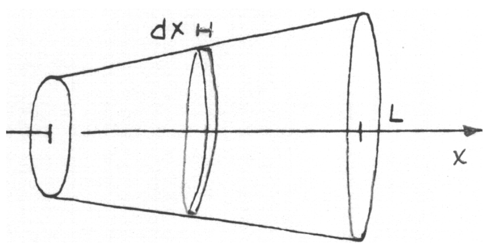
$$q = Q \exp(-t/\tau).$$

The time would be

$$t = \tau \ln \frac{Q}{e}.$$

Because of the ln, even for macroscopic initial charges, the time wouldn't be that large.

3 *Purcell 4.26 Two graphite rods are of equal length. One is a cylinder of radius a. The other is conical, tapering linearly from radius a at one end to radius b at the other. Show that the end-to-end resistance of the conical rod is a/b times that of the cylindrical rod.*



We consider the conical rod to be the series combination of little cylindrical rods of length dx. The radii of these little cylinders are

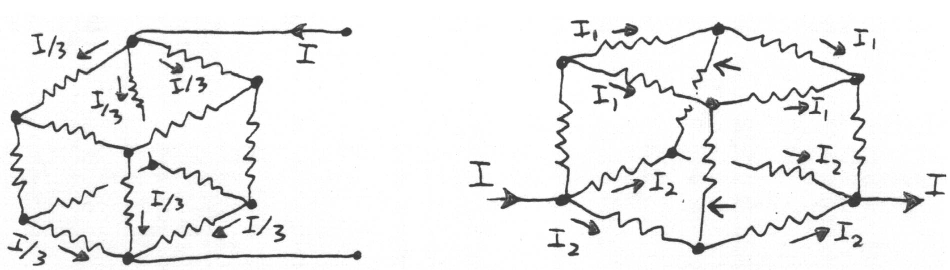
$$r(x) = a + \frac{b-a}{L}x.$$

We sum up the little resistances.

$$R = \int dR = \int_0^L \rho \frac{dx}{A} = \rho \int_0^L \frac{dx}{\pi(a + (b-a)x/L)^2} = \rho \frac{L}{\pi ab} = \frac{a}{b}\left(\rho \frac{L}{\pi a^2}\right)$$

$\rho L/\pi a^2$ is the resistance of the cylinder of radius a.

4 *Purcell 4.31 Suppose a cube has a resistor of resistance R_o along each edge. At each corner the leads from three resistors are soldered together. Find the equivalent resistance between two nodes that represent diagonally opposite corners of the cube. Now find the equivalent resistance between two nodes that correspond to diagonally opposite corners of one face of the cube.*



A total current I enters one node. It then has a choice of three directions to go. Because of the **symmetry**, each choice is identical to the others so the current must split up evenly so that $I/3$ goes through each resistor. Likewise, the current reaching the other node comes through three resistors each having current $I/3$. This leaves 6 resistors in the middle to share the current.

Because each one is identical due to the symmetry, they must each have current $I/6$. To find the voltage drop between the two nodes, follow a straight path from one to the other.

$$V = \frac{I}{3}R_o + \frac{I}{6}R_o + \frac{I}{3}R_o = \frac{5}{6}RI$$

$$R_{eq} = \frac{5}{6}R_o$$

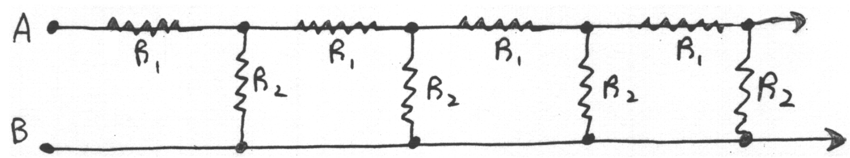
In the second situation, because of the symmetry, we notice that all the resistors on the top square carry the same magnitude of current while all the resistors on the bottom square carry the same magnitude also. This tells us that there is no current through the two resistors indicated by arrows. We can therefore ignore them, because the circuit would behave the same without them. It is then easy to combine the remaining resistors. The top and bottom squares are parallel combinations of resistors $2R_o$.

$$R = \left(\frac{1}{2R_o} + \frac{1}{2R_o}\right)^{-1} = R_o$$

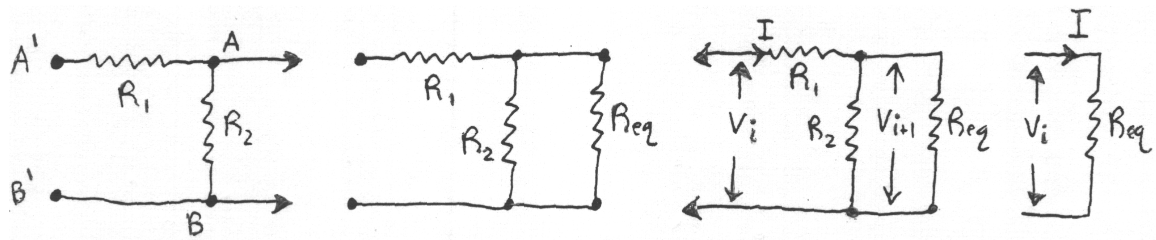
This leaves us with one resistor R_o in parallel with three resistors R_o.

$$R_{eq} = \left(\frac{1}{R_o} + \frac{1}{3R_o}\right)^{-1} = \frac{3}{4}R_o$$

5 *Purcell 4.32 Find the input resistance (between terminals A and B) of the following infinite series.*



Show that, if voltage V_o is applied at the input to such a chain, the voltage at successive nodes decreases in a geometric series. What ratio is required for the resistors to make the ladder an attenuator that halves the voltage at every step? Can you suggest a way to terminate the ladder after a few sections without introducing any error in its attenuation?



If we put another "link" on the left of this infinite chain, we get exactly the same configuration. If this infinite chain has equivalent resistance R_{eq}, the new chain with the extra link can be described by the middle circuit. We can calculate the equivalent resistance of this circuit by considering R_{eq} and R_2 in parallel, in series with R_1. But since this circuit is the same as the original, this equivalent resistance is again R_{eq}.

$$R_{eq} = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_{eq}}\right)^{-1}$$

This leads to the equation

$$R_{eq}^2 - R_1 R_{eq} - R_1 R_2 = 0,$$

with positive solution

$$R_{eq} = \frac{R_1 + \sqrt{R_1^2 + 4R_1 R_2}}{2}.$$

Now consider an arbitrary link with voltage difference V_i between top and bottom. To find the voltage V_{i+1}, we can replace the rest of the series with the equivalent resistor. We could get V_{i+1} if we knew the current through R_1. For this purpose we can replace the ith link also with R_{eq}. The current through this equivalent circuit will also be the current through R_1. This current is just V_i/R_{eq}.

$$V_{i+1} = V_i - R_1 \frac{V_i}{R_{eq}} = V_i \frac{R_{eq} - R_1}{R_{eq}}$$

If we wish to halve the voltage each step,

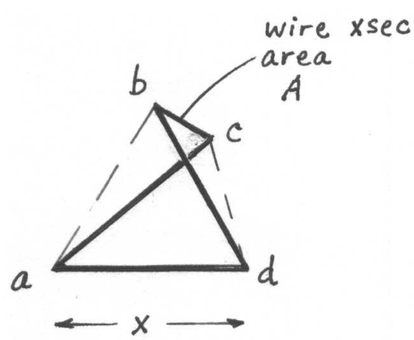
$$\frac{R_1}{R_{eq}} = \frac{1}{2},$$

$$4R_1 = R_1 + \sqrt{R_1^2 + 4R_1 R_2},$$

$$R_2 = 2R_1.$$

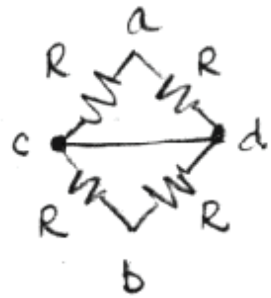
If we wish to terminate the ladder without changing this property, we just replace the rest of the chain at any point with a resistor with resistance R_{eq}.

6 *Extra. Four straight stainless steel wires of length x, cross-sectional area A, and resistivity ρ are welded together so that they lie along four of the six edges of a regular tetrahedron, as shown.*



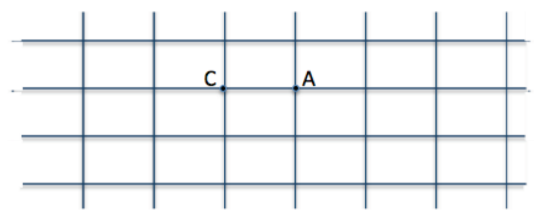
The remaining two sides ab and cd are empty. Consider a and b to be electrical input terminals, and c and d to be electrical output terminals.

When c is shorted to d, what resistance is measured between a and b?



The resitance between a and b is unaffected by short from d to c because all R's are the same. The answer is $(RinparallelwithR) + (RinparallelwithR) = R/2 + R/2 = R$. Where $R = \rho x/A$.

7 *Extra. rectangular wire mesh of infinite extent in a plane has 1 A of current fed into it at a point A, as seen in the diagram and 1 A taken from it at point C. Find the current in the wire AC.*



This problem is solved using **symmetry** and **superposition**. If 1 Ampere is fed into point A and taken out at infinity then from symmetry 1/4 Amps will flow in AC. Likewise if 1 Ampere is taken out from C and fed in at infinity then 1/4 Amps will flow in AC. By superposition of the 2 solutions we obtain the solution of the problem with the current in AC being 1/2 Amps.